

On twisted contact groupoids and on integration of twisted Jacobi manifolds

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Abstract

We introduce the concept of twisted contact groupoids, as an extension either of contact groupoids or of twisted symplectic ones, and we discuss the integration of twisted Jacobi manifolds by twisted contact groupoids. We also investigate the very close relationships which link homogeneous twisted Poisson manifolds with twisted Jacobi manifolds and homogeneous twisted symplectic groupoids with twisted contact ones. Some examples for each structure are presented.

Résumé

On introduit le concept des groupoïdes de contact tordus, comme une extension à la fois des groupoïdes de contact et des groupoïdes symplectiques tordus, et on discute l'intégration des variétés de Jacobi tordues par groupoïdes de contact tordus. En plus, on établit quelques relations étroites qui unissent les variétés de Poisson homogènes tordues avec celles de Jacobi tordues et les groupoïdes homogènes symplectiques tordus avec ceux de contact tordus. Des exemples pour chaque structure étudiée sont présentés.

Keywords: Twisted contact groupoid, homogeneous twisted symplectic groupoid, twisted Jacobi manifold.

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1 Introduction

The notion of *twisted Jacobi manifolds*, introduced by J.M. Nunes da Costa and the author of this present paper in [33], is a weakened version of the notion of *Jacobi manifolds* due to A. Lichnerowicz [29] which, at the same time, generalizes in a natural manner that of *twisted Poisson manifolds* studied by P. Ševera and A. Weinstein [39]. The latter appeared in Park's work on string theory [35], as well as in the work [24] on topological field theory by C. Klimčík and T. Ströbl, and are also relevant in the theory of nonholonomic systems [21]. In fact, the associated bracket on the space of smooth functions on the manifold does not satisfy the Jacobi identity whose failure is controlled by a generalized closed 3-form. Keeping in mind that the Jacobi structures play a central role in the geometric prequantization of Poisson structures [8, 42, 13], one

of the motivations behind the study of twisted Jacobi structures is the likely role that they can play in some geometric prequantization process of twisted Poisson structures.

However, the prequantization of Poisson manifolds is also closely related to the integrability of Jacobi structures [13]. For this reason, the present paper is concerned with the search of the global geometric objects that will integrate twisted Jacobi structures.

The corresponding problem for Jacobi manifolds was treated, independently, by P. Libermann [27], C. Albert [1] and Y. Kerbrat and Z. Souici-Benhammadi [23]. For its approach, they introduced various concepts of *contact groupoids*, as odd-dimensional counterparts of symplectic groupoids [40]. In [17], P. Dazord unified and clarified these concepts and he showed contact groupoids as the central tool for integrating local Lie algebras (it is well known that they are intimately connected with Jacobi structures [19]) and prequantizing Poisson manifolds in Weinstein's sense. While, in [13], M. Crainic and Ch. Zhu discussed this problem using the method of *A*-paths. The analogous question for twisted Poisson manifolds was studied by A.S. Cattaneo and P. Xu [6] and, in the general context of twisted Dirac structures, by H. Bursztyn *et al.* [3]. They presented the notion of *twisted (pre)symplectic groupoids* as a natural extension of the one of symplectic groupoids [40] and they proved that the twisted Poisson manifolds may be regarded as the infinitesimal form of twisted symplectic groupoids.

Inspired by the above studies, we introduce the concept of *twisted contact groupoids*, we establish their fundamental properties (Proposition 3.2), and we prove that these are the global counterparts of twisted Jacobi manifolds (Theorems 3.3 and 4.7). For this, we use the bijection which exists between twisted contact groupoids and *homogeneous twisted symplectic ones* (Proposition 3.5). The latter are a special case of twisted symplectic groupoids and they integrate *homogeneous twisted Poisson manifolds* that are linked to twisted Jacobi manifolds with a very close relationships (Propositions 2.3 and 2.5).

The paper is organized as follows. In Section 2 we review basic definitions and results concerning twisted Jacobi manifolds and homogeneous twisted Poisson ones and we give some examples of twisted contact structures. In Section 3, we introduce the notion of contact groupoids, we describe their basic properties and their connection with the homogeneous twisted symplectic groupoids. Finally, we present some examples of such groupoids. In Section 4, after a brief presentation of the integrability problem of a Lie algebroid and of its solution given by M. Crainic and R.L. Fernandes [10], we prove our main theorem: *An integrable twisted Jacobi manifold is integrated by a twisted contact groupoid*.

Notation: In order to present the concepts of *twisted Jacobi* and *homogeneous twisted Poisson* structures on a smooth manifold M , we recall that, given a bivector field Λ on M , the usual homomorphism of $C^\infty(M, \mathbb{R})$ -modules $\Lambda^\# : \Gamma(T^*M) \rightarrow \Gamma(TM)$, defined, for all $\zeta, \eta \in \Gamma(T^*M)$, by $\langle \eta, \Lambda^\#(\zeta) \rangle = \Lambda(\zeta, \eta)$, can be extended to a homomorphism, also denoted $\Lambda^\#$, from $\Gamma(\bigwedge^k T^*M)$ to $\Gamma(\bigwedge^k TM)$, $k \in \mathbb{N}$, by setting, for any $f \in C^\infty(M, \mathbb{R})$, $\Lambda^\#(f) = f$, and, for all $\zeta \in \Gamma(\bigwedge^k T^*M)$ and $\alpha_1, \dots, \alpha_k \in \Gamma(T^*M)$,

$$\Lambda^\#(\zeta)(\alpha_1, \dots, \alpha_k) = (-1)^k \zeta(\Lambda^\#(\alpha_1), \dots, \Lambda^\#(\alpha_k)). \quad (1)$$

Moreover, for any $(\Lambda, E) \in \Gamma(\bigwedge^2(TM \times \mathbb{R}))$, the homomorphism of $C^\infty(M, \mathbb{R})$ -modules $(\Lambda, E)^\# : \Gamma(T^*M \times \mathbb{R}) \rightarrow \Gamma(TM \times \mathbb{R})$ given, for any $(\zeta, f) \in \Gamma(T^*M \times \mathbb{R})$, by $(\Lambda, E)^\#(\zeta, f) = (\Lambda^\#(\zeta) + fE, -\langle \zeta, E \rangle)$, can be extended to a homomorphism $(\Lambda, E)^\# : \Gamma(\bigwedge^k(T^*M \times \mathbb{R})) \rightarrow \Gamma(\bigwedge^k(TM \times \mathbb{R}))$ by setting, for all $(\zeta, \zeta') \in \Gamma(\bigwedge^k(T^*M \times \mathbb{R}))$ and $(\alpha_1, f_1), \dots, (\alpha_k, f_k) \in \Gamma(T^*M \times \mathbb{R})$,

$$(\Lambda, E)^\#(\zeta, \zeta')((\alpha_1, f_1), \dots, (\alpha_k, f_k)) = (-1)^k (\zeta, \zeta')((\Lambda, E)^\#(\alpha_1, f_1), \dots, (\Lambda, E)^\#(\alpha_k, f_k)).$$

We indicate [22], for any $(X_1, f_1), \dots, (X_k, f_k) \in \Gamma(\bigwedge^k(TM \times \mathbb{R}))$,

$$(\zeta, \zeta')((X_1, f_1), \dots, (X_k, f_k)) = \zeta(X_1, \dots, X_k) + \sum_{i=1}^k (-1)^{i+1} f_i \zeta'(X_1, \dots, \hat{X}_i, \dots, X_k),$$

where the hat denotes missing arguments. Also, following [39], we denote by $(\Lambda^\# \otimes 1)(\zeta)$ the section of $(\bigwedge^{k-1} TM) \otimes T^*M$ that acts on multivector fields by contraction with the factor in T^*M . Precisely, for all $X \in \Gamma(TM)$ and $\alpha_1, \dots, \alpha_{k-1} \in \Gamma(T^*M)$,

$$(\Lambda^\# \otimes 1)(\zeta)(\alpha_1, \dots, \alpha_{k-1})(X) = (-1)^k \zeta(\Lambda^\#(\alpha_1), \dots, \Lambda^\#(\alpha_{k-1}), X).$$

2 Twisted Jacobi and homogeneous twisted Poisson manifolds

A *twisted Jacobi manifold* is a smooth manifold M endowed with a bivector field Λ , a vector field E and a 2-form ω such that

$$\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)} = (\Lambda, E)^\#(d\omega, \omega). \quad (2)$$

The bracket on the left hand side is the Schouten bracket of the Lie algebroid $(TM \times \mathbb{R}, [\cdot, \cdot], \pi)$ over M modified by the 1-cocycle $(0, 1)$ of its Lie algebroid cohomology complex with trivial coefficients [22]. The pair $(d\omega, \omega) \in \Gamma(\bigwedge^3(T^*M \times \mathbb{R}))$ can be viewed as a closed 3-form of $TM \times \mathbb{R}$ with respect to the exterior derivative operator on $\Gamma(\bigwedge(T^*M \times \mathbb{R}))$ defined by $([\cdot, \cdot], \pi)$ and modified by $(0, 1)$ [22]. Writing (2) in terms of the usual Schouten bracket, we obtain [33] its equivalent expression

$$\begin{cases} \frac{1}{2}[\Lambda, \Lambda] + E \wedge \Lambda = \Lambda^\#(d\omega) + \Lambda^\#(\omega) \wedge E \\ [E, \Lambda] = (\Lambda^\# \otimes 1)(d\omega)(E) - ((\Lambda^\# \otimes 1)(\omega)(E)) \wedge E. \end{cases} \quad (3)$$

As explained in [33], the space $C^\infty(M, \mathbb{R})$ of smooth functions on (M, Λ, E, ω) is equipped, just as in the case of ordinary Jacobi manifolds, with the internal composition law

$$\{f, g\} = \Lambda(df, dg) + \langle f dg - g df, E \rangle, \quad f, g \in C^\infty(M, \mathbb{R}), \quad (4)$$

that is bilinear and skew-symmetric but its Jacobi identity acquires an extra term:

$$\{f, \{g, h\}\} + c.p. = (\Lambda, E)^\#(d\omega, \omega)((df, f), (dg, g), (dh, h)), \quad f, g, h \in C^\infty(M, \mathbb{R}). \quad (5)$$

So, (4) is no more a Lie bracket on $C^\infty(M, \mathbb{R})$. However, (Λ, E, ω) produces a Lie algebroid structure $(\{\cdot, \cdot\}^\omega, \pi \circ (\Lambda, E)^\#)$ on the vector bundle $T^*M \times \mathbb{R} \rightarrow M$. The bracket on the space $\Gamma(T^*M \times \mathbb{R})$ of smooth sections of $T^*M \times \mathbb{R}$ is given, for all $(\zeta, f), (\eta, g) \in \Gamma(T^*M \times \mathbb{R})$, by

$$\{(\zeta, f), (\eta, g)\}^\omega = \{(\zeta, f), (\eta, g)\} + (d\omega, \omega)((\Lambda, E)^\#(\zeta, f), (\Lambda, E)^\#(\eta, g), \cdot), \quad (6)$$

$\{\cdot, \cdot\}$ being the Kerbrat-Souici-Benhammadi bracket [23] on $\Gamma(T^*M \times \mathbb{R})$, and the anchor map is $\pi \circ (\Lambda, E)^\#$, where $\pi : TM \times \mathbb{R} \rightarrow TM$ denotes the projection on the first factor. The section $(-E, 0)$ of $TM \times \mathbb{R}$ is a 1-cocycle of the Lie algebroid cohomology complex with trivial coefficients [22] of $(T^*M \times \mathbb{R}, \{\cdot, \cdot\}^\omega, \pi \circ (\Lambda, E)^\#)$. The brackets (4) and (6) are related by

$$\{(df, f), (dg, g)\}^\omega = (d\{f, g\}, \{f, g\}) + (d\omega, \omega)((\Lambda, E)^\#(df, f), (\Lambda, E)^\#(dg, g), \cdot),$$

whence we conclude that the mapping $f \mapsto X_f = \Lambda^\#(df) + fE = \{f, \cdot\} + \langle df, E \rangle$ from functions to their hamiltonian vectors fields is no longer a homomorphism.

If $E = 0$, equations (3) are reduced to $\frac{1}{2}[\Lambda, \Lambda] = \Lambda^\#(d\omega)$, which means that $(\Lambda, d\omega)$ defines an exact twisted Poisson structure [39] on M .

Two characteristic examples of twisted Jacobi manifolds are the following.

Conformal twisted Jacobi manifolds ([33]): Let (M, Λ, E, ω) be a twisted Jacobi manifold and a an element of $C^\infty(M, \mathbb{R})$ that never vanishes on M . We set

$$\Lambda^a = a\Lambda, \quad E^a = \Lambda^\#(da) + aE \quad \text{and} \quad \omega^a = \frac{1}{a}\omega.$$

Then, $(\Lambda^a, E^a, \omega^a)$ defines a new twisted Jacobi structure on M that is called *a-conformal* to the initially given one. Its associated bracket (4) in $C^\infty(M, \mathbb{R})$ is given by

$$\{f, g\}^a = \frac{1}{a}\{af, ag\}, \quad f, g \in C^\infty(M, \mathbb{R}).$$

Twisted contact manifolds ([34]): A *twisted contact manifold* is a $2n+1$ -dimensional smooth manifold M equipped with an 1-form ϑ and a 2-form ω such that $\vartheta \wedge (d\vartheta + \omega)^n \neq 0$, everywhere in M . The Reeb vector field E on M , defined by

$$i(E)\vartheta = 1 \quad \text{and} \quad i(E)(d\vartheta + \omega) = 0, \quad (7)$$

and the bivector field Λ on M whose associated morphism $\Lambda^\#$ is given by

$$\Lambda^\#(\vartheta) = 0 \quad \text{and} \quad i(\Lambda^\#(\zeta))(d\vartheta + \omega) = -(\zeta - \langle \zeta, E \rangle \vartheta), \quad \text{for all } \zeta \in \Gamma(T^*M), \quad (8)$$

yield a $(d\omega, \omega)$ -twisted Jacobi structure on M .

The above definition includes that M is an orientable manifold and $\vartheta \wedge (d\vartheta + \omega)^n$ is a volume form on M . Thus, $\ker \vartheta$ and $\ker(d\vartheta + \omega)$ are complementary subbundles of TM , where $\ker(d\vartheta + \omega)$, called the *vertical bundle*, is of rank 1 and is generated by E , however, $\ker \vartheta = \text{Im} \Lambda^\#$, called the *horizontal bundle* and denoted by \mathcal{H} , is of rank $2n$. If M is not orientable, a twisted contact structure on M cannot be defined by a single pair (ϑ, ω) defined on the whole of M . It is described by an open covering U_i , $i \in I$, of M such that each U_i is endowed with a twisted contact structure (ϑ_i, ω_i) and on the overlaps $U_i \cap U_j$ there exist nowhere vanishing functions f_{ij} satisfying $\vartheta_i = f_{ij}\vartheta_j$ and $\omega_i = f_{ij}\omega_j$. Hence, the family $(U_i, \vartheta_i, \omega_i)$, $i \in I$, defines a locally conformal twisted contact structure on M . Of course, in this case, the above decomposition of TM holds locally.

Remark 2.1 We recall that an *almost cosymplectic structure* [25, 29] on a $2n + 1$ -dimensional smooth manifold M is defined by a pair (ϑ, Θ) , where ϑ is a 1-form and Θ is a 2-form on M , such that $\vartheta \wedge \Theta^n \neq 0$ everywhere on M . So, a twisted contact structure (ϑ, ω) on M can be viewed as an almost cosymplectic for which $\Theta = d\vartheta + \omega$. Reciprocally, each almost cosymplectic manifold (M, ϑ, Θ) can be considered as twisted contact with $\omega = \Theta - d\vartheta$.

Some examples of twisted contact manifolds: In [26], P. Libermann gave a list of examples of almost cosymplectic structures: (i) on the sphere S^5 by embedding it in the almost complex (so, almost symplectic) manifold S^6 and (ii) on the quadrics homeomorphic to $S^1 \times \mathbb{R}^4$, $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$ by embedding they in the quadrics

homeomorphic to $S^2 \times \mathbb{R}^4$ and $S^3 \times \mathbb{R}^3$ which are also endowed with almost complex structures. From Remark 2.1, it is clear that the almost cosymplectic structures obtained on S^5 and on the 5-dimensional quadratics can be viewed as twisted contact.

Let $(M_1, \Lambda_1, E_1, \omega_1)$ and $(M_2, \Lambda_2, E_2, \omega_2)$ be two twisted Jacobi manifolds. A smooth map $\psi : M_1 \rightarrow M_2$ is called *twisted Jacobi map* if it satisfies the following conditions

$$\Lambda_2 = \psi_* \Lambda_1, \quad E_2 = \psi_* E_1 \quad \text{and} \quad \omega_1 = \psi^* \omega_2.$$

Moreover, we say that $\psi : M_1 \rightarrow M_2$ is a *conformal twisted Jacobi map* if there exists a non vanishing smooth function a on M_1 such that ψ is a twisted Jacobi map between $(M_1, \Lambda_1^a, E_1^a, \omega_1^a)$ and $(M_2, \Lambda_2, E_2, \omega_2)$.

A *homogeneous twisted Poisson manifold* [33] is an exact twisted Poisson manifold $(M, \Lambda, d\omega)$ [39],

$$\frac{1}{2}[\Lambda, \Lambda] = \Lambda^\#(d\omega),$$

endowed with a vector field Z , called the *homothety vector field of M* , such that

$$\mathcal{L}_Z \Lambda = -\Lambda \quad \text{and} \quad \omega = i(Z)d\omega. \quad (9)$$

Now, we establish a one to one correspondence between twisted Jacobi and homogeneous twisted Poisson manifolds, as in the non-twisted framework [16].

Proposition 2.2 *Let $(M, \Lambda, d\omega, Z)$ be a homogeneous twisted Poisson manifold and M_0 a 1-codimensional submanifold of M transverse to Z . Then, M_0 receives an induced twisted Jacobi structure $(\Lambda_0, E_0, \omega_0)$ characterized by one of the following properties:*

1. *For any pair (f, g) of homogeneous functions of degree 1 with respect to Z , defined on an open \mathcal{O} of M , the bracket $\{f, g\}$ is also a homogeneous function of degree 1 with respect to Z .*
2. *Let $\varpi : U \rightarrow M_0$ be the projection, along the integral curves of Z , of a tubular neighborhood U of M_0 in M onto M_0 and a a function on U that is equal to 1 on M_0 and homogeneous of degree 1 with respect to Z . Then, the projection ϖ is an a -conformal twisted Jacobi map.*

Proof. We have that any homogeneous function $f \in C^\infty(U, \mathbb{R})$ of degree 1 with respect to Z is of type $f = a\varpi^* f_0$, where $f_0 \in C^\infty(M_0, \mathbb{R})$. Hence, $df = a\varpi^* df_0 + \varpi^* f_0 da$. Let (f, g) be a pair of homogeneous functions of degree 1 with respect to Z , defined on an open $\mathcal{O} \subset U$ of M , and (f_0, g_0) the corresponding pair of functions on M_0 , i.e., $f = a\varpi^* f_0$ and $g = a\varpi^* g_0$. Then,

$$\begin{aligned} \{f, g\} &= \Lambda(df, dg) = \Lambda(a\varpi^* df_0 + \varpi^* f_0 da, a\varpi^* dg_0 + \varpi^* g_0 da) \\ &= a^2 \Lambda(\varpi^* df_0, \varpi^* dg_0) + a\Lambda(\varpi^* df_0, \varpi^* g_0 da) + a\Lambda(\varpi^* f_0 da, \varpi^* dg_0). \end{aligned} \quad (10)$$

Since $\mathcal{L}_Z(a\Lambda) = 0$ and $\mathcal{L}_Z(\Lambda^\#(da)) = 0$, the tensors fields $a\Lambda$ and $\Lambda^\#(da)$ are projectable along the integral curves of Z . Let $\Lambda_0 = \varpi_*(a\Lambda)$ and $E_0 = \varpi_*(\Lambda^\#(da))$ be their projections and $\{\cdot, \cdot\}_0$ the bracket (4) defined by (Λ_0, E_0) . Thus, (10) can be written as

$$\{f, g\} = a\varpi^*(\Lambda_0(df_0, dg_0) + \langle f_0 dg_0 - g_0 df_0, E_0 \rangle) = a\varpi^*\{f_0, g_0\}_0,$$

¹The natural definition of a *homogeneous twisted Poisson manifold* would be a twisted Poisson manifold (M, Λ, φ) [39] with a vector field Z which would satisfy the equations $\mathcal{L}_Z \Lambda = -\Lambda$ and $\mathcal{L}_Z \varphi = \varphi$. But, $\mathcal{L}_Z \varphi = \varphi \Leftrightarrow d(i(Z)\varphi) = \varphi$, whence we get that, in the homogeneous case, φ is exact, i.e., $\varphi = d\omega$ with $\omega = i(Z)\varphi$. Thus, $\omega = i(Z)d\omega \Leftrightarrow \omega = \mathcal{L}_Z \omega$ and $i(Z)\omega = 0$.

which means that $\{f, g\}$ is also homogeneous with respect to Z .

We regard $(\Lambda, d\omega)$ as the twisted Jacobi structure $(\Lambda, 0, \omega)$ on M and we consider its a -conformal structure $(\Lambda^a, E^a, \omega^a)$, $\Lambda^a = a\Lambda$, $E^a = \Lambda^\#(da)$ and $\omega^a = \frac{1}{a}\omega$, which is also twisted Jacobi. Thus, the following equations hold.

$$\begin{cases} \frac{1}{2}[\Lambda^a, \Lambda^a] + E^a \wedge \Lambda^a = (\Lambda^a)^\#(d\omega^a) + (\Lambda^a)^\#(\omega^a) \wedge E^a \\ [E^a, \Lambda^a] = ((\Lambda^a)^\# \otimes 1)(d\omega^a)(E^a) - (((\Lambda^a)^\# \otimes 1)(\omega^a)(E^a)) \wedge E^a. \end{cases} \quad (11)$$

Since $i(Z)d\omega = \omega$ and $\mathcal{L}_Z a = a$, we have $\mathcal{L}_Z \omega^a = 0$ and $i(Z)\omega^a = 0$, which mean that ω^a is projectable along the integral curves of Z . Let ω_0 be the 2-form on M_0 for which $\omega^a = \varpi^* \omega_0$. So, $d\omega^a = \varpi^* d\omega_0$. Also, the equation $[Z, [\Lambda^a, \Lambda^a]] = 0$ means that $[\Lambda^a, \Lambda^a]$ is projectable onto M_0 with respect to Z . Its projection is the Schouten bracket of the projection of Λ^a with itself. Hence, by projecting the system (11) along the integral curves of Z and taking into account that $\Lambda_0 = \varpi_*(a\Lambda)$ and $E_0 = \varpi_*(\Lambda^\#(da))$, we obtain

$$\begin{cases} \frac{1}{2}[\Lambda_0, \Lambda_0] + E_0 \wedge \Lambda_0 = \Lambda_0^\#(d\omega_0) + \Lambda_0^\#(\omega_0) \wedge E_0 \\ [E_0, \Lambda_0] = (\Lambda_0^\# \otimes 1)(d\omega_0)(E_0) - ((\Lambda_0^\# \otimes 1)(\omega_0)(E_0)) \wedge E_0, \end{cases}$$

that signify that $(\Lambda_0, E_0, \omega_0)$ is a twisted Jacobi structure on M_0 and $\varpi : (M, \Lambda, 0, \omega) \rightarrow (M_0, \Lambda_0, E_0, \omega_0)$ is an a -conformal twisted Jacobi map. \blacklozenge

Proposition 2.3 ([33]) *Let (Λ, E, ω) be a twisted Jacobi structure on a smooth manifold M . Its twisted poissonization defines a homogeneous twisted Poisson structure on $\tilde{M} = M \times \mathbb{R}$ whose the bivector field $\tilde{\Lambda}$ and the 2-form $\tilde{\omega}$ are given, respectively, by*

$$\tilde{\Lambda} = e^{-s}(\Lambda + \frac{\partial}{\partial s} \wedge E) \quad \text{and} \quad \tilde{\omega} = e^s \omega,$$

and the homothety vector field is $\frac{\partial}{\partial s}$, where s is the canonical coordinate on \mathbb{R} . The canonical projection $\varpi : M \times \mathbb{R} \rightarrow M$ is a e^s -conformal twisted Jacobi map and the induced twisted Jacobi structure on M , viewed as the submanifold $M \times \{0\}$ of \tilde{M} , in the sense of Proposition 2.2, is the initial given one.

Corollary 2.4 *When the structure (Λ, E) on M comes from a ω -twisted contact form ϑ on M , then $\tilde{\Lambda}$ is the inverse of the twisted symplectic form $\tilde{\Omega} = d(e^s \varpi^* \vartheta) + e^s \varpi^* \omega$ on \tilde{M} , which is homogeneous with respect to $\frac{\partial}{\partial s}$, and reciprocally.*

By applying the above results, we may construct another example of twisted contact manifold.

The cosphere bundle of a twisted Poisson manifold. Let (M, Λ, φ) be a twisted Poisson manifold. As D. Roytenberg has observed [38], the cotangent bundle T^*M of M is endowed with a nondegenerate $d\omega$ -twisted Poisson structure which corresponds to a $d\omega$ -twisted symplectic structure $\Omega = d\vartheta + \omega$, ϑ being the Liouville form on T^*M . In a local coordinates system $(x_1, \dots, x_n, p_1, \dots, p_n)$ of T^*M ,

$$\vartheta = \sum_{i=1}^n p_i dx_i \quad \text{and} \quad \omega = \frac{1}{2} \sum_{i,j,k,l=1}^n p_i \lambda^{ij} \varphi_{jkl} dx_k \wedge dx_l,$$

where λ^{ij} and φ_{jkl} are, respectively, the local components of Λ and φ . We remark that (Ω, ω) is homogeneous with respect to the Liouville vector field $Z = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}$ of T^*M . Consider the action Φ of the multiplicative group $\mathbb{R}_+ = (0, +\infty)$ by dilations

on the fibers of $T^*M \setminus \{0\}$, i.e., for any $x \in M$ and $z \in T_x^*M$, $\Phi(s, z) = sz$. The cosphere bundle S^*M of M is the quotient manifold $(T^*M \setminus \{0\})/\mathbb{R}_+$. By working as in [37], we prove that the cone $S^*M \times \mathbb{R}_+$ over S^*M equipped with the homogeneous twisted symplectic structure $(d(sv) + sw, s\frac{\partial}{\partial s})$, $s \in \mathbb{R}_+$, is twisted symplectomorphic to $(T^*M \setminus \{0\}, \Omega, Z)$. Thus, S^*M receives a twisted contact structure.

We continue with the study of the projection of a twisted Jacobi structure along the integral curves of its vector field.

Proposition 2.5 *Let (M, Λ, E, ω) be a twisted Jacobi manifold and M_0 a submanifold of M , of codimension 1, transverse to E . We denote by $\varpi : U \rightarrow M_0$ the projection onto M_0 of a tubular neighbourhood U of M_0 in M along the integral curves of E , i.e., for any $x_0 \in M_0$, $\varpi^{-1}(x_0)$ is a connected arc of the integral curve of E through x_0 , and by η the 1-form along M_0 that verifies $i(E)\eta = 1$ and $i(X)\eta = 0$, for any vector field X on M tangent to M_0 , and we set $Z_0 = \Lambda^\#(\eta)$. If $\omega = \varpi^*\omega_0$, where ω_0 is a 2-form on M_0 , then, there exists on M_0 a unique exact twisted Poisson structure $(\Lambda_0, d\omega_0)$ such that $\varpi : U \rightarrow M_0$ is a twisted Jacobi map and*

$$\mathcal{L}_{Z_0}\Lambda_0 = -\Lambda_0 - \Lambda_0^\#(d\omega_0(Z_0, \cdot, \cdot) - \omega_0). \quad (12)$$

Moreover, $(\Lambda_0, d\omega_0, Z_0)$ is homogeneous if and only if $\omega_0 = d\omega_0(Z_0, \cdot, \cdot)$.

Proof. Since (Λ, E, ω) defines a twisted Jacobi structure on M , (3) holds on M . Thus, because of $\omega = \varpi^*\omega_0$ and $\varpi_*E = 0$,

$$\begin{aligned} [E, \Lambda] &= (\Lambda^\# \otimes 1)(\varpi^*d\omega_0)(E) - ((\Lambda^\# \otimes 1)(\varpi^*\omega_0)(E)) \wedge E \\ &= -(\varpi_*\Lambda)^\#(d\omega_0(\varpi_*E, \cdot, \cdot)) - (\varpi_*\Lambda)^\#(\omega_0(\varpi_*E, \cdot)) \wedge E = 0, \end{aligned} \quad (13)$$

which means that Λ is projectable along the integral curves of E onto M_0 . Let $\Lambda_0 = \varpi_*\Lambda$ be its projection. Furthermore, (13) implies that $[E, [\Lambda, \Lambda]] = 0$, whence we get that $[\Lambda, \Lambda]$ is also projectable onto M_0 . Its projection $\varpi_*[\Lambda, \Lambda]$ is the Schouten bracket of the projection $\varpi_*\Lambda$ of Λ with itself. Thus, by projecting the first equation of the system (3) parallel to the integral curves of E , we obtain

$$\begin{aligned} \frac{1}{2}[\varpi_*\Lambda, \varpi_*\Lambda] + (\varpi_*E) \wedge (\varpi_*\Lambda) &= \varpi_*(\Lambda^\#(\varpi^*d\omega_0)) + \varpi_*(\Lambda^\#(\varpi^*\omega_0)) \wedge (\varpi_*E) \Leftrightarrow \\ \frac{1}{2}[\Lambda_0, \Lambda_0] &= \Lambda_0^\#(d\omega_0), \end{aligned} \quad (14)$$

whence we deduce that $(\Lambda_0, d\omega_0)$ endows M_0 with an exact twisted Poisson structure and that $\varpi : U \rightarrow M_0$ is a twisted Jacobi map.

Now, by integrating along the integral curves of E and by restricting, if necessary, the tubular neighbourhood U of M_0 in M , we can construct a function h on U such that $h|_{M_0} = 0$ and $i(E)dh = 1$, hence $dh|_{M_0} = \eta$. Let $X_h = \Lambda^\#(dh) + hE$ be the hamiltonian vector field of h with respect to (Λ, E) . Since $[E, \Lambda](dh, \cdot) = 0$, $[E, X_h] = E$, which says that X_h is projectable onto M_0 along the integral curves of E . Let $Z_0 = \varpi_*(X_h) = \varpi_*(\Lambda^\#(dh))$ be its projection which coincides with the restriction $X_h|_{M_0} = \Lambda^\#(\eta)$ of X_h on M_0 .

In order to establish (12), we denote by $\{\cdot, \cdot\}$ the composition law (4) on $C^\infty(M, \mathbb{R})$ determined by (Λ, E) and by $\{\cdot, \cdot\}_0$ the composition law in $C^\infty(M_0, \mathbb{R})$ defined by Λ_0 . Let (f_0, g_0) be a pair of smooth functions on M_0 and $(f, g) = (\varpi^*f_0, \varpi^*g_0)$ the associated pair of smooth functions on M that are constant along the integral curves of E . We have

$$\begin{aligned} \{f, g\} &= \{\varpi^*f_0, \varpi^*g_0\} \stackrel{(4)}{=} \Lambda(\varpi^*df_0, \varpi^*dg_0) + \langle \varpi^*f_0 \varpi^*dg_0 - \varpi^*g_0 \varpi^*df_0, E \rangle \\ &= \varpi^*\{f_0, g_0\}_0. \end{aligned} \quad (15)$$

On the other hand,

$$\begin{aligned}
(X_h - 1)\{f, g\} &= \{h, \{f, g\}\} \\
&\stackrel{(5)}{=} \{\{h, f\}, g\} + \{f, \{h, g\}\} + (\Lambda, E)^\#(d\omega, \omega)((dh, h), (df, f), (dg, g)) \\
&= \{(X_h - 1)f, g\} + \{f, (X_h - 1)g\} + (\Lambda, E)^\#(d\omega, \omega)((dh, h), (df, f), (dg, g)) \tag{16}
\end{aligned}$$

But, for any $f \in C^\infty(M, \mathbb{R})$ of type $f = \varpi^* f_0$ with $f_0 \in C^\infty(M_0, \mathbb{R})$,

$$(X_h - 1)f = \varpi^*(\varpi_*(X_h - 1)f_0) = \varpi^*((Z_0 - 1)f_0). \tag{17}$$

Also,

$$\begin{aligned}
(\Lambda, E)^\#(d\omega, \omega)((dh, h), (df, f), (dg, g)) &\stackrel{(2)}{=} \left(\frac{1}{2}[\Lambda, \Lambda] + E \wedge \Lambda\right)(dh, df, dg) \\
&\quad + h[E, \Lambda](df, dg) + f[E, \Lambda](dg, dh) + g[E, \Lambda](dh, df) \\
&\stackrel{(3)(13)}{=} (\Lambda^\#(d\omega) + \Lambda^\#(\omega) \wedge E)(dh, df, dg) \\
&= \Lambda^\#(\varpi^* d\omega_0)(dh, \varpi^* df_0, \varpi^* dg_0) + \Lambda^\#(\varpi^* \omega_0)(dh, \varpi^* df_0) \langle \varpi^* dg_0, E \rangle \\
&\quad + \Lambda^\#(\varpi^* \omega_0)(\varpi^* df_0, \varpi^* dg_0) \langle dh, E \rangle + \Lambda^\#(\varpi^* \omega_0)(\varpi^* dg_0, dh) \langle \varpi^* df_0, E \rangle \\
&\stackrel{(1)}{=} \varpi^*(-d\omega_0(\varpi_*(\Lambda^\#(dh)), \varpi_*(\Lambda^\#(\varpi^* df_0)), \varpi_*(\Lambda^\#(\varpi^* dg_0)))) \\
&\quad + \varpi^*(\omega_0(\varpi_*(\Lambda^\#(\varpi^* df_0)), \varpi_*(\Lambda^\#(\varpi^* dg_0)))) \\
&= \varpi^*(-d\omega_0(Z_0, \Lambda_0^\#(df_0), \Lambda_0^\#(dg_0)) + \omega_0(\Lambda_0^\#(df_0), \Lambda_0^\#(dg_0))) \\
&\stackrel{(1)}{=} \varpi^*((-\Lambda_0^\#(d\omega_0(Z_0, \cdot, \cdot)) + \Lambda_0^\#(\omega_0))(df_0, dg_0)). \tag{18}
\end{aligned}$$

Thus, taking into account (15), (17) and (18), (16) takes the form

$$\begin{aligned}
\varpi^*((Z_0 - 1)\{f_0, g_0\}_0) &= \{\varpi^*((Z_0 - 1)f_0), \varpi^* g_0\} + \{\varpi^* f_0, \varpi^*((Z_0 - 1)g_0)\} \\
&\quad + \varpi^*((-\Lambda_0^\#(d\omega_0(Z_0, \cdot, \cdot)) + \Lambda_0^\#(\omega_0))(df_0, dg_0)) \stackrel{(15)}{\Leftrightarrow} \\
(Z_0 - 1)\{f_0, g_0\}_0 &= \{(Z_0 - 1)f_0, g_0\}_0 + \{f_0, (Z_0 - 1)g_0\}_0 \\
&\quad + (-\Lambda_0^\#(d\omega_0(Z_0, \cdot, \cdot)) + \Lambda_0^\#(\omega_0))(df_0, dg_0) \Leftrightarrow \\
\mathcal{L}_{Z_0}\{f_0, g_0\}_0 - \{f_0, g_0\}_0 &= \{\mathcal{L}_{Z_0}f_0, g_0\}_0 + \{f_0, \mathcal{L}_{Z_0}g_0\}_0 - 2\{f_0, g_0\}_0 \\
&\quad + (-\Lambda_0^\#(d\omega_0(Z_0, \cdot, \cdot)) + \Lambda_0^\#(\omega_0))(df_0, dg_0) \Leftrightarrow \\
\mathcal{L}_{Z_0}\Lambda_0(df_0, dg_0) &= -\Lambda_0(df_0, dg_0) + (-\Lambda_0^\#(\varphi_0(Z_0, \cdot, \cdot)) + \Lambda_0^\#(\omega_0))(df_0, dg_0),
\end{aligned}$$

whence we deduce (12). Also, according to (9) and (12), we get that $(\Lambda_0, d\omega_0)$ is homogeneous with respect to Z_0 if and only if $\omega_0 = d\omega_0(Z_0, \cdot, \cdot)$. \blacklozenge

3 Twisted contact and homogeneous twisted symplectic groupoids

We recall some basic results about the Lie groupoids which are needed in below and we fix our notations. For more detailed information, we suggest the following standard references [31, 4, 18].

Let $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ be a *Lie groupoid* with *source map* α and *target map* β . We denote by

- $m : \Gamma_2 \rightarrow \Gamma$ the *product map* on the set of composable pairs $\Gamma_2 = \{(g, h) \in \Gamma \times \Gamma / \alpha(g) = \beta(h)\}$ of $\Gamma \times \Gamma$, $(g, h) \mapsto m(g, h) = g \cdot h$, which is compatible with α and β , i.e., for any $(g, h) \in \Gamma_2$, $\alpha(g \cdot h) = \alpha(h)$ and $\beta(g \cdot h) = \beta(g)$, and associative, i.e., for any triple (g, h, k) of composable elements, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$;

- $\varepsilon : \Gamma_0 \hookrightarrow \Gamma$ the *embedding* of Γ_0 into Γ , called the *unit section of Γ* , which associates a unit with each element of Γ_0 ², i.e., for any $g \in \Gamma$, $g \cdot \varepsilon(\alpha(g)) = g = \varepsilon(\beta(g)) \cdot g$, and satisfies $\alpha \circ \varepsilon = \beta \circ \varepsilon = \text{Id}_{\Gamma_0}$;
- $\iota : \Gamma \rightarrow \Gamma$ the *inversion map*, $g \mapsto \iota(g) = g^{-1}$, which has the following properties: for any $g \in \Gamma$, $\alpha(g^{-1}) = \beta(g)$, $\beta(g^{-1}) = \alpha(g)$, $g^{-1} \cdot g = \varepsilon(\alpha(g))$ and $g \cdot g^{-1} = \varepsilon(\beta(g))$.

For each $g \in \Gamma$, the maps $L_g : \beta^{-1}(\alpha(g)) \rightarrow \beta^{-1}(\beta(g))$, $h \mapsto L_g(h) = g \cdot h$, and $R_g : \alpha^{-1}(\beta(g)) \rightarrow \alpha^{-1}(\alpha(g))$, $h \mapsto R_g(h) = h \cdot g$, are smooth diffeomorphisms and they are called the *left translation* and the *right translation* by g , respectively. We denote by $C_L^\infty(\Gamma, \mathbb{R})$ (resp. $C_R^\infty(\Gamma, \mathbb{R})$) the set of *left* (resp. *right*) *invariant functions* on Γ , i.e., the set of smooth functions on Γ that are invariant by the left (resp. right) translations. It is easy to verify that $C_L^\infty(\Gamma, \mathbb{R}) = \alpha^* C^\infty(\Gamma_0, \mathbb{R})$ (resp. $C_R^\infty(\Gamma, \mathbb{R}) = \beta^* C^\infty(\Gamma_0, \mathbb{R})$). Also, we say that a vector field X on Γ is a *left invariant vector field* (resp. *right invariant vector field*) if it is tangent to β -fibers, so $\beta_*(X) = 0$, (resp. tangent to α -fibers, so $\alpha_*(X) = 0$) and invariant under the left translations, i.e., $L_{g*}(X(h)) = X(g \cdot h)$, (resp. the right translations, i.e., $R_{g*}(X(h)) = X(h \cdot g)$).

Let $A(\Gamma) = \ker \beta_* \cap T_{\Gamma_0} \Gamma$ be the vector bundle over Γ_0 consisting of tangent spaces to β -fibers at the points of Γ_0 . By left translations, each section of $A(\Gamma)$ gives rise to a unique left invariant vector field on Γ . Hence, the space of smooth sections $\Gamma(A(\Gamma))$ of $A(\Gamma)$ may be identified with the space of smooth left invariant vector fields on Γ which is closed with respect the usual Lie bracket $[\cdot, \cdot]$ on $T\Gamma$. So, $\Gamma(A(\Gamma))$ inherits a Lie bracket $[\cdot, \cdot]$. $A(\Gamma)$ equipped with the above bracket and the restriction to $A(\Gamma)$ of the linear bundle map $T\alpha : T\Gamma \rightarrow T\Gamma_0$ as anchor map, becomes a Lie algebroid over Γ_0 . $(A(\Gamma), [\cdot, \cdot], T\alpha)$ called the *Lie algebroid of the Lie groupoid* $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$.

Recall that a *Lie algebroid* [31, 18] over a smooth manifold M is a real vector bundle $A \rightarrow M$ together with a Lie bracket $[\cdot, \cdot]$ on the space of smooth sections $\Gamma(A)$ and a bundle map $\rho : A \rightarrow TM$, called the *anchor map*, whose extension to sections satisfies the Leibniz identity

$$[s_1, f s_2] = f[s_1, s_2] + (\rho(s_1)f)s_2, \quad \text{for all } s_1, s_2 \in \Gamma(A) \text{ and } f \in C^\infty(M, \mathbb{R}).$$

Moreover, the presence of a multiplicative function r on a (Lie) groupoid $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$, i.e., for any $(g, h) \in \Gamma_2$, $r(g \cdot h) = r(g) + r(h)$, permits us to define a left groupoid action of Γ on the canonical projection $\varpi : \tilde{\Gamma}_0 = \Gamma_0 \times \mathbb{R} \rightarrow \Gamma_0$ defined by the map

$$\mathbf{ac}^r : \Gamma \star \tilde{\Gamma}_0 \longrightarrow \tilde{\Gamma}_0, \quad \mathbf{ac}^r(g, (x, s)) = (\beta(g), s - r(g)), \quad (19)$$

where $\Gamma \star \tilde{\Gamma}_0 = \{(g, (x, s)) \in \Gamma \times \tilde{\Gamma}_0 / \alpha(g) = \varpi(x, s) = x\}$. Then, $\Gamma \star \tilde{\Gamma}_0$ itself has a (Lie) groupoid structure [31, 18] with manifold of units $\tilde{\Gamma}_0$ and it is called *the action groupoid associated to \mathbf{ac}^r* . It is easy to see that $\Gamma \star \tilde{\Gamma}_0$ may be identified with the product manifold $\tilde{\Gamma} = \Gamma \times \mathbb{R}$. Under this identification, the structural maps $\tilde{\alpha}$ (source) and $\tilde{\beta}$ (target) of $\tilde{\Gamma}$ are given, respectively, by

$$\tilde{\alpha}(g, s) = (\alpha(g), s) \quad \text{and} \quad \tilde{\beta}(g, s) = (\beta(g), s - r(g)), \quad \text{for any } (g, s) \in \tilde{\Gamma}.$$

The set $\tilde{\Gamma}_2$ of composable pairs of $\tilde{\Gamma} \times \tilde{\Gamma}$ is identified with $\Gamma_2 \times \mathbb{R}$ via the map

$$((g_1, s_2 - r(g_2)), (g_2, s_2)) \mapsto ((g_1, g_2), s_2),$$

²For this reason, Γ_0 called, also, manifold of units.

and the multiplication \tilde{m} on $\tilde{\Gamma}_2$ is defined by

$$\tilde{m}((g_1, s_2 - r(g_2)), (g_2, s_2)) = (m(g_1, g_2), s_2).$$

While, the inversion map of $\tilde{\Gamma}$ is $\tilde{\iota}(g, s) = (\iota(g), s - r(g))$ and the unit section of $\tilde{\Gamma}$ is $\tilde{\varepsilon}(x, s) = (\varepsilon(x), s)$.

3.1 Twisted contact groupoids

Definition 3.1 A twisted contact groupoid is a Lie groupoid $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ of dimension $2n + 1$ equipped with a ω -twisted contact form ϑ , ω being a 2-form on Γ of type $\omega = \alpha^* \omega_0 - e^{-r} \beta^* \omega_0$, where $\omega_0 \in \Gamma(\bigwedge^2 T^* \Gamma_0)$ and r is a smooth function on Γ , such that

$$m^* \vartheta = pr_2^*(e^{-r}) \cdot pr_1^* \vartheta + pr_2^* \vartheta \quad (20)$$

holds on the set of composable pairs Γ_2 of $\Gamma \times \Gamma$. In (20), $pr_i : \Gamma_2 \subset \Gamma \times \Gamma \rightarrow \Gamma$ denotes the projection on the i -factor, $i = 1, 2$, of Γ_2 .

In the next proposition we establish the basic properties of a twisted contact groupoid.

Proposition 3.2 Let $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \vartheta, \omega, r)$ be a twisted contact groupoid and $(A(\Gamma), [\cdot, \cdot], \alpha_*)$ its Lie algebroid. We denote by $(\Lambda_\Gamma, E_\Gamma, \omega)$ the $(d\omega, \omega)$ -twisted Jacobi structure on Γ defined by (ϑ, ω) . Then

- i) r is multiplicative, i.e., for any $(g, h) \in \Gamma_2$, $r(g \cdot h) = r(g) + r(h)$, it is annulated on the units of Γ , i.e., $r \circ \varepsilon = 0$, and $r \circ \iota = -r$.
- ii) $\iota^* \vartheta = -e^r \vartheta$.
- iii) Γ_0 , identified with $\varepsilon(\Gamma_0)$, is a Legendrian submanifold of Γ , i.e., $\dim \Gamma_0 = n$ and $\varepsilon^* \vartheta = 0$.
- iv) E_Γ is a left invariant vector field and it has r as first integral.
- v) For any section X_0^l of $A(\Gamma) \rightarrow \Gamma_0$, we write with X^l (resp. X^r) the left (resp. right) invariant vector field on Γ generated by X_0^l (resp. $-\iota_* X_0^l$). Let E_0^l be the section of $A(\Gamma) \rightarrow \Gamma_0$ that corresponds to E_Γ , i.e., $E_\Gamma = E^l$. Then

$$\Lambda_\Gamma^\#(dr) = E^l - e^r E^r. \quad (21)$$

- vi) $\iota : \Gamma \rightarrow \Gamma$ is a $-e^{-r}$ -twisted conformal Jacobi map.
- vii) A vector field Y on Γ is left (resp. right) invariant if and only if there exists $(\zeta_0, f_0) \in \Gamma(T^* \Gamma_0 \times \mathbb{R})$ such that

$$Y = \Lambda_\Gamma^\#(\alpha^* \zeta_0) + \alpha^* f_0 E^l \quad (\text{resp. } Y = e^{-r} \Lambda_\Gamma^\#(\beta^* \zeta_0) + \beta^* f_0 E^r). \quad (22)$$

- viii) For any pair (f_0, g_0) of smooth functions on Γ_0 , $\{\alpha^* f_0, e^{-r} \beta^* g_0\} = 0$, where $\{\cdot, \cdot\}$ denotes the bracket (4) on $C^\infty(\Gamma, \mathbb{R})$ defined by $(\Lambda_\Gamma, E_\Gamma)$.

Proof. i) We consider the tangent groupoid $T\Gamma \xrightarrow[T\beta]{T\alpha} T\Gamma_0$ of $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ [32]. The set of composable pairs in $T\Gamma \times T\Gamma$ is $T\Gamma_2$ with composition law \oplus the tangent map Tm of m , i.e., for any pair $(X, Y) \in T_{(g,h)}\Gamma_2$, where $X = \frac{dg(t)}{dt}|_{t=0} \in T_g\Gamma$, $Y = \frac{dh(t)}{dt}|_{t=0} \in T_h\Gamma$, and $(g(t), h(t))$, $t \in \mathbb{R}$, is a path in Γ_2 with $(g, h) = (g(0), h(0))$,

$$X \oplus Y = T_{(g,h)}m(X, Y) = \frac{d}{dt}(g(t) \cdot h(t))|_{t=0}.$$

From the associativity of Tm and (20) we deduce the multiplicativity of r . Also, for any $g \in \Gamma$, we have $r(g) = r(g \cdot \varepsilon(\alpha(g))) = r(g) + r(\varepsilon(\alpha(g)))$ and $r(g) = r(\varepsilon(\beta(g)) \cdot g) = r(\varepsilon(\beta(g))) + r(g)$, whence, taking into account the surjectivity of the submersions α and β , we get $r \circ \varepsilon = 0$. Furthermore, from the relation $g^{-1} \cdot g = \varepsilon(\alpha(g))$ and the above results, we take $r(g^{-1} \cdot g) = r(\varepsilon(\alpha(g))) \Leftrightarrow r(g^{-1}) + r(g) = 0 \Leftrightarrow (r \circ \iota)(g) = -r(g)$, for any $g \in \Gamma$. Thus, $r \circ \iota = -r$.

ii) For any $g \in \Gamma$ and $X \in \Gamma(T\Gamma)$,

$$X_g = X_g \oplus \varepsilon_*(\alpha_*X)_{\varepsilon(\alpha(g))} = \varepsilon_*(\beta_*X)_{\varepsilon(\beta(g))} \oplus X_g, \quad (23)$$

since $g = g \cdot \varepsilon(\alpha(g)) = \varepsilon(\beta(g)) \cdot g$. So, from the multiplicativity (20) of ϑ , we get

$$\vartheta_g(X_g) = \vartheta_{g \cdot \varepsilon(\alpha(g))}(X_g \oplus \varepsilon_*(\alpha_*X)_{\varepsilon(\alpha(g))}) = e^{-r(\varepsilon(\alpha(g)))}\vartheta_g(X_g) + \vartheta_{\varepsilon(\alpha(g))}(\varepsilon_*(\alpha_*X)_{\varepsilon(\alpha(g))})$$

and

$$\vartheta_g(X_g) = \vartheta_{\varepsilon(\beta(g)) \cdot g}(\varepsilon_*(\beta_*X)_{\varepsilon(\beta(g))} \oplus X_g) = e^{-r(g)}\vartheta_{\varepsilon(\beta(g))}(\varepsilon_*(\beta_*X)_{\varepsilon(\beta(g))}) + \vartheta_g(X_g),$$

whence, taking into consideration of $r \circ \varepsilon = 0$ and of surjectivity of $T\alpha$ and $T\beta$, we obtain $\varepsilon^*\vartheta = 0$. Moreover, from the properties of ι and (20), we get

$$\vartheta_{\varepsilon(\alpha(g))} = \vartheta_{g^{-1} \cdot g} = e^{-r(g)}\vartheta_{g^{-1}} + \vartheta_g \Leftrightarrow 0 = e^{-r(g)}\vartheta_{g^{-1}} + \vartheta_g \Leftrightarrow \vartheta_{g^{-1}} = -e^{r(g)}\vartheta_g, \quad \forall g \in \Gamma,$$

which means that $\iota^*\vartheta = -e^r\vartheta$.

iii) In (ii) we proved $\varepsilon^*\vartheta = 0$. So, in order to establish that $\Gamma_0 \equiv \varepsilon(\Gamma_0)$ is a Legendrian submanifold of Γ , it is enough to show that $\dim \Gamma_0 = n$. We consider the horizontal bundle \mathcal{H} on Γ which is of type $\mathcal{H} = \ker \vartheta$ and, for any $g \in \Gamma$, we denote by \mathcal{H}_g the hyperplane through g . Because $\iota^*\vartheta = -e^r\vartheta$, \mathcal{H} is invariant by inversion. Furthermore, for any $(X, Y) \in \mathcal{H}_g \times \mathcal{H}_h$ with $(g, h) \in \Gamma_2$, $m^*\vartheta_{(g,h)}(X, Y) \stackrel{(20)}{=} e^{-r(h)}\vartheta_g(X) + \vartheta_h(Y) \Leftrightarrow \vartheta_{g \cdot h}(X \oplus Y) = 0 \Leftrightarrow X \oplus Y \in \ker \vartheta_{g \cdot h} = \mathcal{H}_{g \cdot h}$, which means that \mathcal{H} is closed with respect to Tm . Hence, by developing the argumentation of P. Dazord [17] for the pair $(\mathcal{H}, d\vartheta + \omega)$, we get $\dim \Gamma_0 = n$.

iv) At every point of $\varepsilon(\Gamma_0) \equiv \Gamma_0$, the vector field E_Γ can be decomposed as $E_\Gamma = E_0^l + \varepsilon_*(\beta_*(E_\Gamma))$, where E_0^l is the tangent component to the β -fibre through the considered point. Let E^l be the left invariant vector field on Γ generated by E_0^l . We have, at each $g \in \Gamma$, $E_g^l = 0_g \oplus E_{\varepsilon(\alpha(g))}^l$. Therefore, $\vartheta_g(E_g^l) = \vartheta_{g \cdot \varepsilon(\alpha(g))}(0_g \oplus E_{\varepsilon(\alpha(g))}^l) \stackrel{(20)}{=} e^{-r(\varepsilon(\alpha(g)))}\vartheta_g(0_g) + \vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l) = \vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l)$. But, $1 = \vartheta_{\varepsilon(\alpha(g))}(E_{\Gamma \varepsilon(\alpha(g))}) = \vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l + \varepsilon_*(\beta_*(E_\Gamma))_{\varepsilon(\alpha(g))}) = \vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l)$. Consequently,

$$\vartheta_g(E_g^l) = 1. \quad (24)$$

By exterior differentiation of (20), we obtain

$$m^*d\vartheta = pr_2^*de^{-r} \wedge pr_1^*\vartheta + pr_2^*e^{-r} \cdot pr_1^*d\vartheta + pr_2^*d\vartheta, \quad (25)$$

and we apply (25) to the pair $(E_g^l, E_{\Gamma g})$ of vectors at the point $g = g \cdot \varepsilon(\alpha(g))$. Because $r \circ \varepsilon = 0$, $E_g^l = 0_g \oplus E_{\varepsilon(\alpha(g))}^l$ and $E_{\Gamma g} = E_{\Gamma g} \oplus \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}$, we take

$$\begin{aligned} d\vartheta_g(E_g^l, E_{\Gamma g}) &= de^{-r(\varepsilon(\alpha(g)))}(E_{\varepsilon(\alpha(g))}^l) \vartheta_g(E_{\Gamma g}) + d\vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_* E_\Gamma)_{\varepsilon(\alpha(g))}) \stackrel{(7)}{\Leftrightarrow} \\ \omega_g(E_{\Gamma g}, E_g^l) &= de^{-r(\varepsilon(\alpha(g)))}(E_{\varepsilon(\alpha(g))}^l) + d\vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_* E_\Gamma)_{\varepsilon(\alpha(g))}). \end{aligned} \quad (26)$$

We compare $d\vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))})$ with $d\vartheta_g(E_g^l, E_{\Gamma g})$. From (24) and the fact $\varepsilon^* \vartheta = 0$, we get

$$d\vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}) = -\vartheta_{\varepsilon(\alpha(g))}([E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}])$$

and

$$d\vartheta_g(E_g^l, E_{\Gamma g}) = -\vartheta_g([E_g^l, E_{\Gamma g}]).$$

But,

$$\begin{aligned} [E_g^l, E_{\Gamma g}] &= [0_g \oplus E_{\varepsilon(\alpha(g))}^l, E_{\Gamma g} \oplus \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}] \\ &= [0_g, E_{\Gamma g}] \oplus [E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}]. \end{aligned} \quad (27)$$

So, $\vartheta_g([E_g^l, E_{\Gamma g}]) \stackrel{(20)(27)}{=} \vartheta_{\varepsilon(\alpha(g))}([E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}])$, which yields

$$d\vartheta_{\varepsilon(\alpha(g))}(E_{\varepsilon(\alpha(g))}^l, \varepsilon_*(\alpha_*(E_\Gamma))_{\varepsilon(\alpha(g))}) = d\vartheta_g(E_g^l, E_{\Gamma g}). \quad (28)$$

Consequently, (26) can be written as

$$(d\vartheta_g + \omega_g)(E_{\Gamma g}, E_g^l) = de^{-r(\varepsilon(\alpha(g)))}(E_{\varepsilon(\alpha(g))}^l) \cdot \vartheta_g(E_{\Gamma g}), \quad (29)$$

whence, taking into account (7), we deduce that $de^{-r(\varepsilon(\alpha(g)))}(E_{\varepsilon(\alpha(g))}^l) = 0$. Hence, from the last equation and (29), we take

$$(d\vartheta_g + \omega_g)(E_g^l, \cdot) = -de^{-r(\varepsilon(\alpha(g)))}(E_{\varepsilon(\alpha(g))}^l) \cdot \vartheta_g = 0,$$

which means that E_g^l and $E_{\Gamma g}$ are parallel vectors of $T_g \Gamma$. Also, we have $\vartheta_g(E_{\Gamma g}) \stackrel{(7)}{=} 1 \stackrel{(24)}{=} \vartheta_g(E_g^l)$. Hence, we conclude that, for any $g \in \Gamma$, $E_{\Gamma g} = E_g^l$, which means that E_Γ is left invariant.

Next, we will show that r is a first integral of E_Γ . At every point of $\varepsilon(\Gamma_0) \equiv \Gamma_0$, the vector field E_Γ can be decomposed as $E_\Gamma = E_0^r + \varepsilon_*(\alpha_*(E_\Gamma))$, where E_0^r is the tangent component to the α -fibre through the considered point. Let E^r be the right invariant vector field on Γ generated by E_0^r . Because, for any $g \in \Gamma$, $g = \varepsilon(\beta(g)) \cdot g$ and $E_g^r = E_{\varepsilon(\beta(g))}^r \oplus 0_g$, $\vartheta_g(E_g^r) = \vartheta_{\varepsilon(\beta(g)) \cdot g}(E_{\varepsilon(\beta(g))}^r \oplus 0_g) \stackrel{(20)}{=} e^{-r(g)} \vartheta_{\varepsilon(\beta(g))}(E_{\varepsilon(\beta(g))}^r)$. But, $1 \stackrel{(7)}{=} \vartheta_{\varepsilon(\beta(g))}(E_{\Gamma \varepsilon(\beta(g))}) = \vartheta_{\varepsilon(\beta(g))}(E_{\varepsilon(\beta(g))}^r)$. So, $\vartheta_g(E_g^r) = e^{-r(g)}$. Also,

$$\begin{aligned} d\vartheta_g(E_g^r, E_{\Gamma g}) &= d\vartheta_{\varepsilon(\beta(g)) \cdot g}(E_{\varepsilon(\beta(g))}^r \oplus 0_g, 0_{\varepsilon(\beta(g))} \oplus E_{\Gamma g}) \stackrel{(7)(25)}{\Leftrightarrow} \\ \omega_g(E_{\Gamma g}, E_g^r) &= -de^{-r(g)}(E_{\Gamma g}) \cdot \vartheta_{\varepsilon(\beta(g))}(E_{\varepsilon(\beta(g))}^r) \Leftrightarrow \\ \omega_{0\alpha(g)}(\alpha_*(E_\Gamma)_{\alpha(g)}, \alpha_*(E^r)_{\alpha(g)}) - \omega_{0\beta(g)}(\beta_*(E_\Gamma)_{\beta(g)}, \beta_*(E^r)_{\beta(g)}) &= -de^{-r(g)}(E_{\Gamma g}) \Leftrightarrow \\ 0 &= -de^{-r(g)}(E_{\Gamma g}), \end{aligned}$$

whence we conclude $\mathcal{L}_{E_\Gamma} r = 0$.

v) We consider the hamiltonian vector field $X_{e^{-r}} = \Lambda_\Gamma^\#(de^{-r}) + e^{-r}E_\Gamma$ of e^{-r} and the right invariant vector field E^r on Γ defined in the proof of (v). We apply (25) to the pair $(E^r, X_{e^{-r}})$ at a point $g = \varepsilon(\beta(g)) \cdot g$ of Γ and we prove, as in (iv), that $X_{e^{-r}} = E^r$. Since $E_\Gamma = E^l$ at the points of $\varepsilon(\Gamma_0) \equiv \Gamma_0$, we have $E_0^l = E_0^r + \varepsilon_*(\alpha_*(E_0^l))$. Thus, $0 = \beta_*(E_0^r + \varepsilon_*(\alpha_*(E_0^l))) = \beta_*(E_0^r) + \alpha_*(E_0^l) = \beta_*(E_0^r) + \beta_*(\iota_*E_0^l)$. Because $E_0^r + \iota_*E_0^l$ is tangent to α -fibers, the above equation yields that $E_0^r + \iota_*E_0^l = 0$, whence we deduce that E^r coincide with the right invariant vector field on Γ generated by $-\iota_*E_0^l$. Hence, (21) holds.

vi) It is enough to prove that $\iota_*(-e^{-r}\Lambda_\Gamma) = \Lambda_\Gamma$, $\iota_*X_{-e^{-r}} = E_\Gamma$ and $\iota^*\omega = -e^r\omega$. Since $\alpha \circ \iota = \beta$, $\beta \circ \iota = \alpha$ and $r \circ \iota = -r$, we have $\iota^*\omega = \iota^*(\alpha^*\omega_0 - e^{-r}\beta^*\omega_0) = \beta^*\omega_0 - e^{-r\circ\iota}\alpha^*\omega_0 = -e^r\omega$. Also, $\iota_*X_{-e^{-r}} = \iota_*(-E^r) = E^l$ and, for any $\zeta \in T_g^*\Gamma$,

$$\begin{aligned} & \iota^*[i((T_g\iota(-e^{-r(g)}\Lambda_{\Gamma g})^\#{}^tT_g\iota)(\zeta))(d\vartheta + \omega)_{g^{-1}}] = i((-e^{-r(g)}\Lambda_{\Gamma g})^\#(\iota^*\zeta))\iota^*[(d\vartheta + \omega)_{g^{-1}}] \\ & = i((-e^{-r(g)}\Lambda_{\Gamma g})^\#(\iota^*\zeta))[-e^{r(g)}(d\vartheta + \omega)_g - e^{r(g)}dr \wedge \vartheta_g] \\ & = i(\Lambda_{\Gamma g}^\#(\iota^*\zeta))(d\vartheta + \omega)_g + \Lambda_{\Gamma g}(\iota^*\zeta, dr)\vartheta_g \\ & \stackrel{(8)(21)}{=} -\iota^*\zeta + \langle \iota^*\zeta, E_{\Gamma g} \rangle \vartheta_g - \langle \iota^*\zeta, E_g^l \rangle \vartheta_g + \langle \iota^*\zeta, e^{r(g)}E_g^r \rangle \vartheta_g \\ & \stackrel{(ii)}{=} -\iota^*\zeta - \langle \zeta, \iota_*E_g^r \rangle \iota^*(\vartheta_{g^{-1}}) = \iota^*[-\zeta + \langle \zeta, E_{\Gamma g^{-1}} \rangle \vartheta_{g^{-1}}] = \iota^*[i(\Lambda_{\Gamma g^{-1}}^\#(\zeta))(d\vartheta + \omega)_{g^{-1}}], \end{aligned}$$

whence we obtain $T\iota \circ (-e^{-r}\Lambda_\Gamma)^\# \circ {}^tT\iota = \Lambda_\Gamma^\#$, since $\mathcal{H} = \text{Im}\Lambda_\Gamma^\#$ is invariant by ι .

vii) We establish (22) as in Proposition 4.3 of [17]. For the proof we use the pair $(\mathcal{H}, d\vartheta + \omega)$ and the fact that $\omega = \alpha^*\omega_0 - e^{-r}\beta^*\omega_0$.

viii) We consider a pair (f_0, g_0) of smooth functions on Γ_0 and we compute the bracket $\{\alpha^*f_0, e^{-r}\beta^*g_0\}$. Since $T\beta \circ \Lambda_\Gamma^\# \circ {}^tT\alpha = 0$ (see [17]), $\mathcal{L}_{E_\Gamma}r = 0$ and $\beta_*(E_\Gamma) = 0$, we have

$$\{\alpha^*f_0, e^{-r}\beta^*g_0\} = e^{-r}\beta^*g_0 \langle \alpha^*df_0, \Lambda_\Gamma^\#(dr) - E_\Gamma \rangle.$$

But, in (v) we proved that $\Lambda_\Gamma^\#(dr) - E_\Gamma = -e^rX_{e^{-r}}$ and $X_{e^{-r}}$ is right invariant. So, $\alpha_*(\Lambda_\Gamma^\#(dr) - E_\Gamma) = 0$ and $\{\alpha^*f_0, e^{-r}\beta^*g_0\} = 0$. The last equation means that the twisted contact hamiltonian vector fields corresponding to the left invariant functions of Γ have as first integrals the right invariant functions of Γ multiplied by e^{-r} , and reciprocally. \blacklozenge

Now, we can formulate the following theorem.

Theorem 3.3 *Let $(\Gamma \xrightarrow[\beta]{} \Gamma_0, \vartheta, \omega, r)$, $\omega = \alpha^*\omega_0 - e^{-r}\beta^*\omega_0$ with $\omega_0 \in \Gamma(\bigwedge^2 T^*\Gamma_0)$, be a twisted contact groupoid, $(A(\Gamma), [\cdot, \cdot], T\alpha)$ its associated Lie algebroid, and $(\Lambda_\Gamma, E_\Gamma, \omega)$ the twisted Jacobi structure on Γ defined by the ω -twisted contact form ϑ . Then*

- 1) $(\Lambda_\Gamma, E_\Gamma, \omega)$ induces on Γ_0 a twisted Jacobi structure $(\Lambda_0, E_0, \omega_0)$ such that $\alpha : \Gamma \rightarrow \Gamma_0$ is a twisted Jacobi map and $\beta : \Gamma \rightarrow \Gamma_0$ is a $-e^{-r}$ -conformal twisted Jacobi map.
- 2) The map

$$\begin{aligned} \mathfrak{I} : \Gamma(T^*\Gamma_0 \times \mathbb{R}) & \longrightarrow \Gamma(A(\Gamma)) \\ (\zeta_0, f_0) & \mapsto \Lambda_\Gamma^\#(\alpha^*\zeta_0) + \alpha^*f_0E^l \end{aligned}$$

induces a Lie algebroid isomorphism between $(T^*\Gamma_0 \times \mathbb{R}, \{\cdot, \cdot\}^{\omega_0}, \pi \circ (\Lambda_0, E_0)^\#)$, which is the Lie algebroid over Γ_0 determined by $(\Lambda_0, E_0, \omega_0)$, and $(A(\Gamma), [\cdot, \cdot], T\alpha)$.

Proof. 1) Let (f_0, g_0) be a pair of smooth functions on Γ_0 and $(\alpha^*f_0, \alpha^*g_0)$ the corresponding pair of left invariant functions on Γ . We have

$$\{\alpha^*f_0, \alpha^*g_0\} = \Lambda_\Gamma(\alpha^*df_0, \alpha^*dg_0) + \langle \alpha^*f_0\alpha^*dg_0 - \alpha^*g_0\alpha^*df_0, E_\Gamma \rangle.$$

Obviously, the function $\Lambda_\Gamma(\alpha^*df_0, \alpha^*dg_0)$ is left invariant. On the other hand, since E_Γ is left invariant, it acts by differentiation on $C_L^\infty(\Gamma, \mathbb{R}) = \alpha^*C^\infty(\Gamma_0, \mathbb{R})$ [4]. Thus, $\langle \alpha^*f_0\alpha^*dg_0 - \alpha^*g_0\alpha^*df_0, E_\Gamma \rangle$ is also a left invariant function on Γ . So, $\{\alpha^*f_0, \alpha^*g_0\}$ is left invariant. Consequently, there exists $h_0 \in C^\infty(\Gamma_0, \mathbb{R})$, $h_0 = \varepsilon^*\{\alpha^*f_0, \alpha^*g_0\}$, such that $\{\alpha^*f_0, \alpha^*g_0\} = \alpha^*h_0$. The map $(f_0, g_0) \mapsto h_0$ endows $C^\infty(\Gamma_0, \mathbb{R})$ with an internal composition law $\{\cdot, \cdot\}_0$ which is the bracket (4) defined by a twisted Jacobi structure $(\Lambda_0, E_0, \omega_0)$ on Γ_0 . Effectively, for any $u_0, v_0 \in C^\infty(\Gamma_0)$,

$$\begin{aligned} \{\alpha^*u_0, \alpha^*v_0\} &\stackrel{(4)}{=} \Lambda_\Gamma(\alpha^*du_0, \alpha^*dv_0) + \langle \alpha^*u_0\alpha^*dv_0 - \alpha^*v_0\alpha^*du_0, E_\Gamma \rangle \Leftrightarrow \\ \alpha^*\{u_0, v_0\}_0 &= \alpha^*(\Lambda_0(du_0, dv_0) + \langle u_0dv_0 - v_0du_0, E_0 \rangle), \end{aligned}$$

where Λ_0 is the bivector field on Γ_0 associated to $\Lambda_0^\# = T\alpha \circ \Lambda_\Gamma^\# \circ {}^tT\alpha$ and $E_0 = \alpha_*(E_\Gamma)$. Since α is a submersion surjective, the last equation means that $\{\cdot, \cdot\}_0$ is the bracket (4) defined by (Λ_0, E_0) . Moreover, for any $u_0, v_0, w_0 \in C^\infty(\Gamma_0)$,

$$\begin{aligned} \{\alpha^*u_0, \{\alpha^*v_0, \alpha^*w_0\}\} + c.p. &\stackrel{(5)(3)}{=} (\Lambda_\Gamma^\#(d\omega) + \Lambda_\Gamma^\#(\omega) \wedge E_\Gamma)(\alpha^*du_0, \alpha^*dv_0, \alpha^*dw_0) \\ &+ \alpha^*u_0((\Lambda_\Gamma^\# \otimes 1)(d\omega)(E_\Gamma) - (\Lambda_\Gamma^\# \otimes 1)(\omega)(E_\Gamma) \wedge E_\Gamma)(\alpha^*dv_0, \alpha^*dw_0) + c.p. \quad (30) \end{aligned}$$

The left side of (30) is equal to $\alpha^*(\{u_0, \{v_0, w_0\}_0\}_0 + c.p.)$. However, taking into consideration that $\omega = \alpha^*\omega_0 - e^{-r}\beta^*\omega_0$, $\beta_*E_\Gamma = 0$ and $\mathcal{L}_{E_\Gamma}r = 0$, the right side of (30) can be written as

$$\begin{aligned} &\alpha^*((\Lambda_0^\#(d\omega_0) + \Lambda_0^\#(\omega_0) \wedge E_0)(du_0, dv_0, dw_0)) \\ &+ [\alpha^*u_0\alpha^*((\Lambda_0^\# \otimes 1)(d\omega_0)(E_0) - (\Lambda_0^\# \otimes 1)(\omega_0)(E_0) \wedge E_0)(dv_0, dw_0)] + c.p. \end{aligned}$$

From the above results and the surjectivity of the submersion α , we deduce that, for any $u_0, v_0, w_0 \in C^\infty(\Gamma_0, \mathbb{R})$,

$$\begin{aligned} \{u_0, \{v_0, w_0\}_0\}_0 + c.p. &= (\Lambda_0^\#(d\omega_0) + \Lambda_0^\#(\omega_0) \wedge E_0)(du_0, dv_0, dw_0) \\ &+ u_0((\Lambda_0^\# \otimes 1)(d\omega_0)(E_0) - (\Lambda_0^\# \otimes 1)(\omega_0)(E_0) \wedge E_0)(dv_0, dw_0) + c.p., \end{aligned}$$

which means that $(\Lambda_0, E_0, \omega_0)$ defines a twisted Jacobi structure on Γ_0 . By construction, α is a twisted Jacobi map.

Since α is a twisted Jacobi map and ι is a $-e^{-r}$ -conformal twisted Jacobi map (see Proposition 3.2), their composition $\beta = \alpha \circ \iota$ is a $-e^{-r}$ -conformal twisted Jacobi map.

2) In order to show that $\mathfrak{I} : \Gamma(T^*\Gamma_0 \times \mathbb{R}) \longrightarrow \Gamma(A(\Gamma))$ induces a Lie algebroid isomorphism, it suffices to prove that \mathfrak{I} is compatible with the brackets and the anchors maps and that $\ker \mathfrak{I} = (0, 0)$. We denote also by α^* the map from $\Gamma(T^*\Gamma_0 \times \mathbb{R})$ to $\Gamma(T^*\Gamma \times \mathbb{R})$ given, for any $(\zeta_0, f_0) \in \Gamma(T^*\Gamma_0 \times \mathbb{R})$, by $\alpha^*(\zeta_0, f_0) = (\alpha^*\zeta_0, \alpha^*f_0)$. Being $E_\Gamma = E^l$, we have $\mathfrak{I} = \pi \circ (\Lambda_\Gamma, E_\Gamma)^\# \circ {}^tT\alpha$. Since α is a twisted Jacobi map, $T\beta \circ \Lambda_\Gamma^\# \circ {}^tT\alpha = 0$ and E_Γ is left invariant, it is easy to check that α^* is a Lie algebras homomorphism, i.e., for all $(\zeta_0, f_0), (\eta_0, g_0) \in \Gamma(T^*\Gamma_0 \times \mathbb{R})$,

$$\alpha^*(\{(\zeta_0, f_0), (\eta_0, g_0)\}^{\omega_0}) = \{\alpha^*(\zeta_0, f_0), \alpha^*(\eta_0, g_0)\}^\omega. \quad (31)$$

Thus, by applying the anchor map $\pi \circ (\Lambda_\Gamma, E_\Gamma)^\#$ to (31), we obtain

$$\begin{aligned} \mathfrak{I}(\{(\zeta_0, f_0), (\eta_0, g_0)\}^{\omega_0}) &= (\pi \circ (\Lambda_\Gamma, E_\Gamma)^\#)\{\alpha^*(\zeta_0, f_0), \alpha^*(\eta_0, g_0)\}^\omega \\ &= [(\pi \circ (\Lambda_\Gamma, E_\Gamma)^\#)(\alpha^*(\zeta_0, f_0)), (\pi \circ (\Lambda_\Gamma, E_\Gamma)^\#)(\alpha^*(\eta_0, g_0))] \\ &= [\mathfrak{I}((\zeta_0, f_0)), \mathfrak{I}((\eta_0, g_0))], \end{aligned}$$

which signifies that \mathfrak{I} is compatible with the brackets on $\Gamma(T^*\Gamma_0 \times \mathbb{R})$ and $\Gamma(A(\Gamma))$. As well, we have $T\alpha \circ \mathfrak{I} = T\alpha \circ \pi \circ (\Lambda_\Gamma, E_\Gamma)^\# \circ {}^t T\alpha = \pi \circ T\alpha \circ (\Lambda_\Gamma, E_\Gamma)^\# \circ {}^t T\alpha = \pi \circ (\Lambda_0, E_0)^\#$, whence we get the compatibility of \mathfrak{I} with the anchors of $T^*\Gamma_0 \times \mathbb{R}$ and $A(\Gamma)$. Moreover, since $\ker \vartheta = \text{Im} \Lambda_\Gamma^\#$ and $\ker(d\vartheta + \omega) = \langle E^l \rangle$ are complementary subbundles of $T\Gamma$ and α is a submersion surjective, $\mathfrak{I}(\zeta_0, f_0) = \Lambda_\Gamma^\#(\alpha^*\zeta_0) + \alpha^*f_0E^l = 0$ if and only if $(\zeta_0, f_0) = (0, 0)$. So, $\ker \mathfrak{I} = (0, 0)$. \blacklozenge

3.2 Homogeneous twisted symplectic and twisted contact groupoids

We complete this section by investigating, in the groupoid framework, the relationship which links homogeneous twisted symplectic structures with twisted contact structures. For this reason, we recall the following concepts.

A *multiplicative vector field on a Lie groupoid* $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ [31] is a pair of vectors fields (Z, Z_0) , where $Z \in \Gamma(T\Gamma)$ and $Z_0 \in \Gamma(T\Gamma_0)$, such that $Z : \Gamma \rightarrow T\Gamma$ is a morphism of Lie groupoids over $Z_0 : \Gamma_0 \rightarrow T\Gamma_0$, when $T\Gamma \xrightarrow[\beta]{\alpha} T\Gamma_0$ is endowed with the tangent Lie groupoid structure. The above condition is equivalent to the property that the flows Ψ_t of Z are (local) Lie groupoid automorphisms over the flows ψ_t of Z_0 .

A *twisted symplectic groupoid* [6] is a Lie groupoid $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ equipped with a nondegenerate multiplicative 2-form Ω on Γ , i.e., $m^*\Omega = pr_1^*\Omega + pr_2^*\Omega$, where $pr_i : \Gamma_2 \rightarrow \Gamma$ is the projection on the i -factor, $i = 1, 2$, of Γ_2 , and a closed 3-form ϕ_0 on Γ_0 such that $d\Omega = \alpha^*\phi_0 - \beta^*\phi_0$. If the closed 3-form ϕ_0 is exact, i.e., $\phi_0 = d\omega_0$ with $\omega_0 \in \Gamma(\bigwedge^2 T^*\Gamma_0)$, we say that $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \Omega, d\omega_0)$ is an *exact twisted symplectic groupoid*. Then, Ω has the particular type $\Omega = \alpha^*\omega_0 - \beta^*\omega_0 + d\eta$, where η is an 1-form on Γ , and it is multiplicative if and only if $d\eta$ is.

Taking into consideration the first footnote, we introduce the notion of a *homogeneous twisted symplectic groupoid* as follows.

Definition 3.4 A homogeneous twisted symplectic groupoid is an exact twisted symplectic groupoid $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \Omega, d\omega_0)$ endowed with a multiplicative vector field (Z, Z_0) such that $\mathcal{L}_Z \Omega = \Omega$ and $i(Z_0)d\omega_0 = \omega_0$.

Hence, we have

Proposition 3.5 If $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \vartheta, \omega, r)$, $\omega = \alpha^*\omega_0 - e^{-r}\beta^*\omega_0$ with $\omega_0 \in \Gamma(\bigwedge^2 T^*\Gamma_0)$, is a $2n+1$ -dimensional twisted contact groupoid, then its action groupoid $\tilde{\Gamma} \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} \tilde{\Gamma}_0$, which is defined by (19), endowed with the structure $(\tilde{\Omega}, d\tilde{\omega}_0, (\frac{\partial}{\partial s}, \frac{\partial}{\partial s}))$, where $\tilde{\Omega} = d(e^s\vartheta) + e^s\omega$, $\tilde{\omega}_0 = e^s\omega_0$ and s is the canonical coordinate on \mathbb{R} , is a homogeneous twisted symplectic groupoid, and reciprocally.

Proof. By construction, $\dim \tilde{\Gamma} = 2n+2$. So, $\tilde{\Omega}$ is of maximal rank on $\tilde{\Gamma}$, i.e., $\tilde{\Omega}^{2n+2} \neq 0$, if and only if $\vartheta \wedge (d\vartheta + \omega)^n \neq 0$, i.e., (ϑ, ω) is a twisted contact structure on Γ . Furthermore, $d\tilde{\Omega} = d(e^s\omega) = d(e^s(\alpha^*\omega_0 - e^{-r}\beta^*\omega_0)) = d(e^s\alpha^*\omega_0) - d(e^{s-r}\beta^*\omega_0) = \tilde{\alpha}^*(d\tilde{\omega}_0) - \tilde{\beta}^*(d\tilde{\omega}_0)$ and the multiplicativity of $\tilde{\Omega}$ is equivalent with that of (ϑ, r) . Effectively, let

$\tilde{p}r_i$ be the projection on the i -factor, $i = 1, 2$, of $\tilde{\Gamma}_2$ and $((g_1, s_2 - r(g_2)), (g_2, s_2))$ an element of $\tilde{\Gamma}_2$. Since $\tilde{\Omega} = e^s ds \wedge \vartheta + e^s d\vartheta + e^s \omega$, we have

$$\tilde{m}^* \tilde{\Omega}_{((g_1, s_2 - r(g_2)), (g_2, s_2))} = \tilde{\Omega}_{(m(g_1, g_2), s_2)} = e^{s_2} [ds \wedge \vartheta_{m(g_1, g_2)} + d\vartheta_{m(g_1, g_2)} + \omega_{m(g_1, g_2)}] \quad (32)$$

and

$$\begin{aligned} (\tilde{p}r_1^* \tilde{\Omega} + \tilde{p}r_2^* \tilde{\Omega})_{((g_1, s_2 - r(g_2)), (g_2, s_2))} &= \tilde{\Omega}_{(g_1, s_2 - r(g_2))} + \tilde{\Omega}_{(g_2, s_2)} \\ &= e^{s_2 - r(g_2)} d(s - r) \wedge \vartheta_{g_1} + e^{s_2 - r(g_2)} (d\vartheta_{g_1} + \omega_{g_1}) \\ &\quad + e^{s_2} ds \wedge \vartheta_{g_2} + e^{s_2} d\vartheta_{g_2} + e^{s_2} \omega_{g_2}. \end{aligned} \quad (33)$$

Hence, $\tilde{m}^* \tilde{\Omega}_{((g_1, s_2 - r(g_2)), (g_2, s_2))} = (\tilde{p}r_1^* \tilde{\Omega} + \tilde{p}r_2^* \tilde{\Omega})_{((g_1, s_2 - r(g_2)), (g_2, s_2))}$ if and only if the components of (32) and (33) containing ds are equals and as also ones containing ω . The first equality gives us $\vartheta_{m(g_1, g_2)} = e^{-r(g_2)} \vartheta_{g_1} + \vartheta_{g_2}$, whence we obtain (20). From the second equality, taking into account that $\alpha(g_1 \cdot g_2) = \alpha(g_2)$, $\beta(g_1 \cdot g_2) = \beta(g_1)$ and $\alpha(g_1) = \beta(g_2)$, we get $\omega_{m(g_1, g_2)} = e^{-r(g_2)} \omega_{g_1} + \omega_{g_2} \Leftrightarrow r(g_1 \cdot g_2) = r(g_1) + r(g_2)$.

Finally, it is enough to show that $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ is a multiplicative vector field on $\tilde{\Gamma} \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} \tilde{\Gamma}_0$

and $(\tilde{\Omega}, \tilde{\omega}_0)$ is homogeneous with respect to $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$. By considering the identifications $T\tilde{\Gamma} \equiv T\Gamma \times T\mathbb{R}$ and $T\tilde{\Gamma}_0 \equiv T\Gamma_0 \times T\mathbb{R}$, we can easily check that $T\tilde{\alpha} \circ \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \circ \tilde{\alpha}$, $T\tilde{\beta} \circ \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \circ \tilde{\beta}$ and $T\tilde{m}(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = \frac{\partial}{\partial s} \circ \tilde{m}$. Hence, $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ is a multiplicative vector field on $\tilde{\Gamma}$. The homogeneity of $(\tilde{\Omega}, \tilde{\omega}_0)$ with respect to $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ is obvious. \blacklozenge

Moreover, by adapting Theorem 2.6 of A. Cattaneo and P. Xu [6] in the homogeneous case, we obtain

Theorem 3.6 *Let $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \Omega, d\omega_0, (Z, Z_0))$ be a homogeneous twisted symplectic groupoid, $(A(\Gamma), [\cdot, \cdot], T\alpha)$ its associated Lie algebroid, and $(\Lambda, d\omega, Z)$, $\omega = \alpha^* \omega_0 - \beta^* \omega_0$, the homogeneous twisted Poisson structure on Γ defined by $(\Omega, d\omega)$. Then*

- 1) $(\Lambda, d\omega, Z)$ induces on Γ_0 a homogeneous twisted Poisson structure $(\Lambda_0, d\omega_0, Z_0)$ such that $\alpha : \Gamma \rightarrow \Gamma_0$ is a twisted Poisson map and $\beta : \Gamma \rightarrow \Gamma_0$ is a twisted anti-Poisson map.
- 2) If $(T^*\Gamma_0, [\cdot, \cdot]^{d\omega_0}, \Lambda_0^\#)$ is the Lie algebroid over Γ_0 determined by $(\Lambda_0, d\omega_0)$, i.e., for any $\zeta_0, \eta_0 \in \Gamma(T^*\Gamma_0)$,

$$[\zeta_0, \eta_0]^{d\omega_0} = \mathcal{L}_{\Lambda_0^\#(\zeta_0)} \eta_0 - \mathcal{L}_{\Lambda_0^\#(\eta_0)} \zeta_0 - d\Lambda_0(\zeta_0, \eta_0) + d\omega_0(\Lambda_0^\#(\zeta_0), \Lambda_0^\#(\eta_0), \cdot),$$

the map $\mathfrak{J} : \Gamma(T^*\Gamma_0) \longrightarrow \Gamma(A(\Gamma))$, $\zeta_0 \mapsto \Lambda^\#(\alpha^* \zeta_0)^3$, induces a Lie algebroid isomorphism between $(T^*\Gamma_0, [\cdot, \cdot]^{d\omega_0}, \Lambda_0^\#)$ and $(A(\Gamma), [\cdot, \cdot], T\alpha)$.

Proof. Taking into account the results of [6] and our data $i(Z_0)d\omega_0 = \omega_0$, we have only to check the homogeneity of Λ_0 with respect to Z_0 , where $\Lambda_0 = \alpha_* \Lambda = -\beta_* \Lambda$. Since Λ is the inverse of Ω and $\mathcal{L}_Z \Omega = \Omega$, we get $\mathcal{L}_Z \Lambda = -\Lambda$. Also, from the multiplicativity of (Z, Z_0) on Γ , we have $T\alpha \circ Z = Z_0 \circ \alpha$ and $T\beta \circ Z = Z_0 \circ \beta$. So, because α is a submersion surjective, $\mathcal{L}_{Z_0} \Lambda_0 = \mathcal{L}_{\alpha_* Z} (\alpha_* \Lambda) = \alpha_* (\mathcal{L}_Z \Lambda) = \alpha_* (-\Lambda) = -\Lambda_0$. \blacklozenge

Using Proposition 3.5 and Theorem 3.6, we can formulate the following

³Using the same argumentation as in the case of symplectic groupoids [9], we can show that the left invariant vectors fields on a twisted symplectic groupoid $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \Omega, \phi_0)$ is of type $\Lambda^\#(\alpha^* \zeta_0)$, where $\zeta_0 \in \Gamma(T^*\Gamma_0)$.

Proposition 3.7 *Let $(\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0, \vartheta, \omega, r)$ be a twisted contact groupoid, $(\Lambda_0, E_0, \omega_0)$ its induced twisted Jacobi structure on Γ_0 (Theorem 3.3) and $(\tilde{\Gamma} \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} \tilde{\Gamma}_0, \tilde{\Omega}, d\tilde{\omega}_0, (\frac{\partial}{\partial s}, \frac{\partial}{\partial s}))$ its associated homogeneous twisted symplectic groupoid whose the induced homogeneous twisted Poisson structure on $\tilde{\Gamma}_0$ (Theorem 3.6) is $(\tilde{\Lambda}_0, d\tilde{\omega}_0, \frac{\partial}{\partial s})$. Then, the twisted Poissonization of $(\Lambda_0, E_0, \omega_0)$ coincides with $(\tilde{\Lambda}_0, d\tilde{\omega}_0, \frac{\partial}{\partial s})$.*

3.3 Examples of twisted contact and homogeneous twisted symplectic groupoids

1. A twisted contact groupoid constructed by a twisted contact manifold.

Let $(\Gamma_0, \vartheta_0, \omega_0)$ be a $2n + 1$ -dimensional, simply connected, twisted contact manifold and (Λ_0, E_0) the corresponding twisted Jacobi structure on Γ_0 characterized by (7) and (8). We consider the pair groupoid $\Gamma_0 \times \Gamma_0 \xrightarrow[p_1]{p_2} \Gamma_0$ of Γ_0 , $p_i, i = 1, 2$, being the projection

on the i -factor, and the vector bundle groupoid $\mathbb{R} \xrightarrow[\pi]{\pi} \{0\}$. By identifying $\Gamma_0 \times \{0\}$ with Γ_0 , we construct the product groupoid $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ of $\Gamma_0 \times \Gamma_0$ with \mathbb{R} , i.e., $\Gamma = \Gamma_0 \times \Gamma_0 \times \mathbb{R}$.

The structure maps α and β of Γ are, respectively, the projections on the second and first factor of Γ , that are $\alpha(x, y, t) = y$ and $\beta(x, y, t) = x$. The multiplication map m on $\Gamma_2 = \{(x, y, t), (y, z, s) / x, y, z \in \Gamma_0 \text{ and } t, s \in \mathbb{R}\} \subset \Gamma \times \Gamma$ is given by $m((x, y, t), (y, z, s)) = (x, z, t + s)$, the inversion map $\iota : \Gamma \rightarrow \Gamma$ by $\iota(x, y, t) = (y, x, -t)$ and the embedding $\varepsilon : \Gamma_0 \hookrightarrow \Gamma$ by $\varepsilon(x) = (x, x, 0)$. The triple (ϑ, ω, r) , where

$$\vartheta = \alpha^* \vartheta_0 - e^{-r} \beta^* \vartheta_0, \quad \omega = \alpha^* \omega_0 - e^{-r} \beta^* \omega_0 \quad \text{and} \quad r = p_3,$$

p_3 being the projection $\Gamma \rightarrow \mathbb{R}$, defines a r -multiplicative twisted contact structure on Γ whose the Reeb vector field is $E_\Gamma = 0 + E_0 + 0$ and its bivector field (8) is $\Lambda_\Gamma = -e^r \Lambda_0 + \Lambda_0 + 0$. It is easy to verify that

$$\vartheta \wedge (d\vartheta + \omega)^{2n+1} = ce^{-(n+1)r} \alpha^*(\vartheta_0 \wedge (d\vartheta_0 + \omega_0)^n) \wedge \beta^*(\vartheta_0 \wedge (d\vartheta_0 + \omega_0)^n) \wedge dr \neq 0 \quad (34)$$

on Γ , where c is a nonzero constant dependent of n , and that (ϑ, r) satisfies (20). Clearly, the induced twisted Jacobi structure on Γ_0 by $(\Lambda_\Gamma, E_\Gamma, \omega)$ is the initial given one.

2. The gauge groupoid of a principal line bundle over a twisted contact manifold.

Let $(\Gamma_0, \vartheta_0, \omega_0)$ be a $2n + 1$ -dimensional twisted contact manifold, (Λ_0, E_0) the corresponding twisted Jacobi structure on Γ_0 defined by (7) and (8), and f_0 a first integral of E_0 . Let, also, $\pi : P \rightarrow \Gamma_0$ be a principal line bundle over Γ_0 , i.e., its structure group G is a 1-dimensional Lie group that acts freely on P on the right and $P/G = \Gamma_0$, endowed with a connection \mathfrak{H} . Precisely, \mathfrak{H} is a G -invariant distribution on P , complementary to its vertical bundle $\mathfrak{V} = \ker \pi_*$. We consider the gauge groupoid $\Gamma \xrightarrow[\beta]{\alpha} \Gamma_0$ of P which is the quotient $P \times P/G \rightrightarrows P/G$ of the pair groupoid $P \times P \xrightarrow[\pi_1]{\pi_2} P$ of P , $\pi_i, i = 1, 2$, being the projection on the i -copy of P , by the right diagonal action of G on $P \times P$. The natural projection $\varpi : P \times P \rightarrow P \times P/G$, $(p_1, p_2) \xrightarrow{\varpi} [(p_1, p_2)] = \{(p_1 g, p_2 g) / g \in G\}$, is a Lie groupoid morphism over π . Thus, the source map α and the target map β of Γ are, respectively, the maps defined by $\alpha \circ \varpi = \pi \circ \pi_2$ and $\beta \circ \varpi = \pi \circ \pi_1$. Hence, for each $[(p_1, p_2)] \in \Gamma$, $\alpha([(p_1, p_2)]) = \pi(p_2)$ and $\beta([(p_1, p_2)]) = \pi(p_1)$, while the inversion $\iota : \Gamma \rightarrow \Gamma$ and the embedding $\varepsilon : \Gamma_0 \hookrightarrow \Gamma$ are

given, respectively, by $\iota([(p_1, p_2)]) = [(p_2, p_1)]$ and $\varepsilon(x) = [(p, p)]$, where $p \in \pi^{-1}(x)$. Moreover, the product of two composable elements $[(p_1, p_2)], [(q_1, q_2)]$ of Γ , that means that $\alpha([(p_1, p_2)]) = \pi(p_2) = \pi(q_1) = \beta([(q_1, q_2)])$, is $m([(p_1, p_2)], [(q_1, q_2)]) = [(p_1, q_2g)]$, where g is the unique element of G for which $p_2 = q_1g$. The triple (ϑ, ω, r) , where

$$\vartheta = \alpha^* \vartheta_0 - e^{-r} \beta^* \vartheta_0, \quad \omega = \alpha^* \omega_0 - e^{-r} \beta^* \omega_0 \quad \text{and} \quad r = \alpha^* f_0 - \beta^* f_0,$$

defines a r -multiplicative twisted contact structure on Γ , i.e., (34) and (20) hold on Γ and Γ_2 , respectively. In order to present the tensors fields Λ_Γ and E_Γ that define the corresponding $(d\omega, \omega)$ -twisted Jacobi structure on Γ , we recall that given a q -vector field Q_0 on Γ_0 there exists a unique q -vector field Q_0^h on P , called the *horizontal lift of Q_0 on P with respect to \mathfrak{H}* , such that, for any $p \in P$, $Q_0^h(p) \in \bigwedge^q \mathfrak{H}_p$ and Q_0^h is π -related to Q_0 , i.e.,

$$Q_0^h(\pi^* \zeta_1, \dots, \pi^* \zeta_q) = Q_0(\zeta_1, \dots, \zeta_q) \circ \pi, \quad \text{for all } \zeta_1, \dots, \zeta_q \in \Gamma(T^* \Gamma_0).$$

Because of the G -invariance of \mathfrak{H} , any q -vector field of type $Q_0^h + Q_0'^h$ on $P \times P$ descends to the quotient $P \times P/G$. Hence, we may easily verify that

$$\Lambda_\Gamma = \varpi_*(-e^{\varpi^* r} \Lambda_0^h + \Lambda_0^h) \quad \text{and} \quad E_\Gamma = \varpi_*(0 + E_0^h).$$

Obviously, the induced twisted Jacobi structure on Γ_0 by $(\Lambda_\Gamma, E_\Gamma, \omega)$ is the initial given one.

3. An homogeneous twisted symplectic groupoid. Let (S, ω) be a symplectic manifold, $\Pi = \Pi^\#(\omega)$ the corresponding nondegenerate Poisson tensor and $(\Gamma_0, t\Pi)$, $\Gamma_0 = S \times \mathbb{R}$ and t being the canonical coordinate on \mathbb{R} , the associated Heisenberg-Poisson manifold [41] to S . We suppose that $(\Gamma_0, t\Pi)$ is integrable [41] and we denote by $(\Gamma, \Omega) \xrightarrow[\beta]{\alpha} \Gamma_0$ the symplectic groupoid which integrates it. Since $t\Pi$ is an exact

Poisson structure [18] (there is the vector field $T_0 = -t \frac{1}{t}$ which satisfies $t\Pi = [t\Pi, T_0]$), the multiplicative symplectic form Ω on Γ is also exact [12]. Precisely, the flow of T_0 integrates to the flow of a vector field T on Γ such that $\Omega = d(i(T)\Omega)$ [12]. So, $\Omega = d\sigma$, where $\sigma = i(T)\Omega$, and it is homogeneous with respect to T .

Now, we consider on Γ_0 a twisted Poisson structure (Λ, φ) such that Λ and φ are projectable along the integral curves of T_0 . (For example, we can take $\Lambda = f\Pi$ and $\varphi = -f^{-2}df \wedge \omega$ with f a nonconstant function on S .) Let Ω_0 be the corresponding nondegenerate twisted symplectic 2-form on $T^* \Gamma_0$ [38, 7] which, in a local coordinate system $(x_1, \dots, x_n, p_1, \dots, p_n)$, $n = \dim \Gamma_0$, of $T^* \Gamma_0$, is written as

$$\Omega_0 = \sum_{i=1}^n dp_i \wedge dx_i + \frac{1}{2} \sum_{i,j,k,l=1}^n p_i \lambda^{ij} \varphi_{jkl} dx_k \wedge dx_l,$$

where λ^{ij} and φ_{jkl} are, respectively, the local components of Λ and φ . We remark that Ω_0 is homogeneous with respect to the Liouville vector field $Z_0 = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}$ on $T^* \Gamma_0$.

On the other hand, we consider a left Lie groupoid action of Γ on the canonical projection $q : T^* \Gamma_0 \rightarrow \Gamma_0$ with action map $\Phi : \Gamma \times T^* \Gamma_0 \rightarrow T^* \Gamma_0$, where $\Gamma \times T^* \Gamma_0 = \{(g, z) \in \Gamma \times T^* \Gamma_0 / \alpha(g) = q(z)\}$, compatible with the structure maps of Γ , [18]. Let $\Gamma \ltimes T^* \Gamma_0 \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} T^* \Gamma_0$ be the semi-direct product groupoid of Γ with $T^* \Gamma_0$, [18]. We have

$\Gamma \ltimes T^* \Gamma_0 = \Gamma \star T^* \Gamma_0$, $\tilde{\alpha}(g, z) = z$, $\tilde{\beta}(g, z) = \Phi(g, z)$, $\tilde{\iota}(g, z) = (g^{-1}, \Phi(g, z))$, $\tilde{\varepsilon}(z) = (\varepsilon(q(z)), z)$ and $\tilde{m}((g, z), (g', z')) = (m(g, g'), z')$ with $z = \Phi(g', z')$. The multiplicative nondegenerate 2-form

$$\tilde{\Omega} = \tilde{\alpha}^* \omega_0 - \tilde{\beta}^* \omega_0 + d\tilde{\eta},$$

where

$$\omega_0 = \frac{1}{2} \sum_{i,j,k,l=1}^n p_i \lambda^{ij} \varphi_{jkl} dx_k \wedge dx_l \quad \text{and} \quad \tilde{\eta} = \tilde{\alpha}^* \left(\sum_{i=1}^n p_i dx_i \right) - \tilde{\beta}^* \left(\sum_{i=1}^n p_i dx_i \right) + \sigma,$$

endows $\Gamma \ltimes T^*\Gamma_0$ with a $(\tilde{\alpha}^* d\omega_0 - \tilde{\beta}^* d\omega_0)$ -twisted symplectic structure which is homogeneous with respect to the multiplicative vector field $(T + (T_0^* + Z_0), T_0^* + Z_0)$ ⁴ on $\Gamma \ltimes T^*\Gamma_0 \Rightarrow T^*\Gamma_0$. In above, T_0^* denotes the vector field on $T^*\Gamma_0$ which, in the local coordinate system $(x_1, \dots, x_n, p_1, \dots, p_n)$ of $T^*\Gamma_0$, is written as $T_0^* = T_0 + \sum_{i=1}^n 0 \frac{\partial}{\partial p_i}$.

4 Integration of twisted Jacobi structures

In this section we study the inverse problem of that which was presented in Theorem 3.3. Specifically, we discuss the following question: *Given a twisted Jacobi manifold (M, Λ, E, ω) , does there exist a twisted contact groupoid, with manifold of units M , such that its associated Lie algebroid is isomorphic to $(T^*M \times \mathbb{R}, \{\cdot, \cdot\}^\omega, \pi \circ (\Lambda, E)^\#)$?* When this is the case, we say that (M, Λ, E, ω) is *integrable*.

The above problem is a special case of the problem of integration of a Lie algebroid which asks for the existence of a Lie groupoid whose Lie algebroid is isomorphic to a given one. It was a longstanding problem of Differential Geometry which was solved, in 2003, by M. Crainic and R.L. Fernandes [10]. However, the preceded works of J. Pradines [36] and R. Almeida with P. Molino [2] for Lie algebroids integrated by local Lie groupoids, K. Mackenzie [30] for transitive Lie algebroids, P. Dazord [14] and P. Dazord with G. Hector [15] for the Lie algebroids structures on the cotangent bundles of regular Poisson manifolds, and A.S. Cattaneo with G. Felder [5] for the Lie algebroids structures on the cotangent bundles of arbitrary Poisson manifolds, had also an important contribution in its study.

In specific, M. Crainic and R.L. Fernandes [10] have proved that with each given Lie algebroid $(A, [\cdot, \cdot], \rho)$ over a smooth manifold M , we can associate a topological groupoid $G(A)$, called the *Weinstein groupoid of A* , with simply connected β -fibres⁵, which is the quotient of the space of A -paths in A by an homotopy equivalence relation, and they have obtained necessary and sufficient conditions under which $G(A)$ is a Lie groupoid. Whenever this holds, the Lie algebroid of $G(A)$ is isomorphic to A .

Next, we will describe, in brief, the construction of the Weinstein groupoid $G(A)$ for a given Lie algebroid $(A, [\cdot, \cdot], \rho)$, and some properties of $G(A)$ acquired from the presence on A of an 1-cocycle of its Lie algebroid cohomology complex with trivial coefficients.

⁴We recall that the space tangent to $\Gamma \ltimes T^*\Gamma_0$ at a point (g, z) is the vector subspace of $T_{(g,z)}(\Gamma \ltimes T^*\Gamma_0) = T_g\Gamma \times T_z T^*\Gamma_0$, $\{(X, Y) \in T_g\Gamma \times T_z T^*\Gamma_0 / \alpha_* X(\alpha(g)) = q_* Y(q(z))\}$ (see [28], p. 345). Hence, we have that $T_g + (T_0^* + Z_0)_z$ is tangent to $\Gamma \ltimes T^*\Gamma_0$ at the point (g, z) because $\alpha_* T(\alpha(g)) = T_{0\alpha(g)} = q_*(T_0^* + Z_0)(q(z))$. Also, we have $T\tilde{\alpha} \circ (T + (T_0^* + Z_0))(g, z) = (T_0^* + Z_0)_z = (T_0^* + Z_0) \circ \tilde{\alpha}(g, z)$ and $T\tilde{\beta} \circ (T + (T_0^* + Z_0))(g, z) = \Phi_*(T + (T_0^* + Z_0))(\Phi(g, z)) = (T_0^* + Z_0)_{\Phi(g,z)} = (T_0^* + Z_0) \circ \beta(g, z)$, whence we obtain the multiplicativity of $(T + (T_0^* + Z_0), T_0^* + Z_0)$. The seconde equation of the above sequence is deduced from the compatibility condition $q \circ \Phi = \beta \circ p_1$ of Φ with β , where p_1 is the projection of $\Gamma \ltimes T^*\Gamma_0$ on Γ .

⁵We defined the Lie algebroid $A(\Gamma)$ of a Lie groupoid $\Gamma \xrightarrow[\beta]{} \Gamma_0$ as $A(\Gamma) = \ker \beta_* \cap T_{\Gamma_0}\Gamma$, while

M. Crainic and R.L. Fernandes in [10] identified $A(\Gamma)$ with $\ker \alpha_* \cap T_{\Gamma_0}\Gamma$. For this reason, we must construct $G(A)$ by changing the roles of α and β in the construction of $G(A)$ presented in [10]. On the other hand, we know that the inversion map of a Lie groupoid sends α -fibers to β -fibers, and reciprocally. So, if the α -fibers are simply connected, then the β -fibers are also simply connected, and reciprocally.

4.1 The Weinstein groupoid

Definition 4.1 ([10]) *Let $\pi : A \rightarrow M$ be a Lie algebroid and $\rho : A \rightarrow TM$ its anchor map. An A -path of A is a smooth map $c : I = [0, 1] \rightarrow A$ of class C^1 such that, for all $t \in I$,*

$$\rho(c(t)) = \frac{d\gamma(t)}{dt},$$

where $\gamma(t) = \pi \circ c(t)$ is a path in M and it is called the base path of c . The set of all A -paths of A is denoted by $P(A)$.

Definition 4.2 ([10]) *An A -connection on a vector bundle E over M is a bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(A)$, $(\zeta, \xi) \mapsto \nabla_\zeta \xi$, such that, for any $f \in C^\infty(M, \mathbb{R})$ and $(\zeta, \xi) \in \Gamma(A) \times \Gamma(E)$,*

$$\nabla_{f\zeta} \xi = f \nabla_\zeta \xi \quad \text{and} \quad \nabla_\zeta (f\xi) = f \nabla_\zeta \xi + \mathcal{L}_{\rho(\zeta)}(f)\xi.$$

If $E = A$, we define the torsion of the A -connection ∇ on A as the $C^\infty(M, \mathbb{R})$ -bilinear map $T_\nabla : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ given, for any $\zeta_1, \zeta_2 \in \Gamma(A)$, by

$$T_\nabla(\zeta_1, \zeta_2) = \nabla_{\zeta_1} \zeta_2 - \nabla_{\zeta_2} \zeta_1 - [\zeta_1, \zeta_2].$$

We note that with each standard TM -connection ∇ on A , we can associate an obvious A -connection on A , denoted also by ∇ , by setting, for any $\zeta_1, \zeta_2 \in \Gamma(A)$, $\nabla_{\zeta_1} \zeta_2 \equiv \nabla_{\rho(\zeta_1)} \zeta_2$. Let ∇ be a such connection on A and ∂_t the induced derivative operator which associates to each A -path $c : I \rightarrow A$ the path $\partial_t c$ in A which is the ∇ -horizontal component of $\frac{dc}{dt}$, i.e., if $\gamma : I \rightarrow M$ is the base path of c and ζ is a time dependent section of A such that $\zeta(t, \gamma(t)) = c(t)$, then

$$\partial_t c = \nabla_{\rho(c(t))} \zeta + \frac{d\zeta}{dt}.$$

Definition 4.3 ([10]) *An A -homotopy is a family $c_\epsilon : I \rightarrow A$, $c_\epsilon(t) = c(\epsilon, t)$, of A -paths which depends on a parameter $\epsilon \in I$ in a C^2 -fashion and which has the following properties: (i) their base paths $\gamma_\epsilon : I \rightarrow M$, $\gamma_\epsilon(t) = \gamma(\epsilon, t)$, have fixed ends points and (ii) the solution $b(\epsilon, t)$ of the equation*

$$\partial_t b - \partial_\epsilon c = T_\nabla(c, b), \quad b(\epsilon, 0) = 0,$$

satisfies, for all $\epsilon \in I$, $b(\epsilon, 1) = 0$.⁶

Two A -paths c_0 and c_1 are said to be homotopic, and we write $c_0 \sim c_1$, if there exists an A -homotopy $c : I \times I \rightarrow A$ joining them, i.e., $c(0, t) = c_0(t)$ and $c(1, t) = c_1(t)$.

For a given A -path c , we denote by $[c]$ the set of all homotopic A -paths to c .

We define now the Weinstein groupoid $G(A) \xrightarrow[\beta]{\alpha} M$ of $\pi : A \rightarrow M$ as the space of classes of homotopic paths of $P(A)$, i.e.,

$$G(A) := P(A) / \sim,$$

with the following structure maps. For any $[c] \in G(A)$, the source map $\alpha : G(A) \rightarrow M$ and the target map $\beta : G(A) \rightarrow M$ are given, respectively, by $\alpha([c]) = \gamma(0)$ and $\beta([c]) = \gamma(1)$, where γ is the base path of c , while the inversion $\iota : G(A) \rightarrow G(A)$ maps $[c]$ to the class $[\bar{c}]$ of the opposite A -path \bar{c} of c , i.e., for any $t \in [0, 1]$, $\bar{c}(t) = -c(1-t)$,

⁶We recall that the solution b does not depend on ∇ , [10].

and the embedding $\varepsilon : M \hookrightarrow G(A)$ maps each point x of M to the class $[0_x]$ of the constant trivial path above x . The set $G_2(A)$ of composable pairs of $G(A) \times G(A)$ consists of the pairs of classes $([c_1], [c_0])$ whose representatives (c_1, c_0) are pairs of composable A -paths, i.e., $\pi(c_1(0)) = \pi(c_0(1))$, and the product map m on $G_2(A)$ is given by $m([c_1], [c_0]) = [c_1] \cdot [c_0] = [c_1 \odot c_0]$, where $c_1 \odot c_0$ is the concatenation of (c_0, c_1) ⁷ defined as follows

$$c_1 \odot c_0(t) \equiv \begin{cases} 2c_0(2t), & 0 \leq t \leq \frac{1}{2}, \\ 2c_1(2t-1), & \frac{1}{2} < t \leq 1. \end{cases}$$

Theorem 4.4 ([10]) *The groupoid $G(A)$ is a β -simply connected topological groupoid. Moreover, whenever A is integrable, $G(A)$ admits a smooth structure which makes it into the unique β -simply connected Lie groupoid integrating A .*

As M. Crainic and R.L. Fernandes have shown [10], $G(A)$ can be viewed as the leaf space of a foliation $\mathcal{F}(A)$ on $P(A)$, of finite codimension, whose leaves are just the classes of homotopic A -paths. In particular, $\mathcal{F}(A)$ may be defined by the orbits of a Lie algebra action on the space $\tilde{P}(A)$ of all C^1 curves $c : I \rightarrow A$ with base path $\gamma = \pi \circ c$ of class C^2 , which contains $P(T^*M)$ as submanifold. An alternative description of $G(A)$, when A is the Lie algebroid T^*M of a twisted Poisson manifold M , is developed by A.S. Cattaneo and P. Xu [6]. By modifying the method introduced in [5] for Poisson manifolds, they have obtained $G(T^*M)$ by an appropriate symplectic reduction of the cotangent bundle $T^*P(M)$ of the space $P(M)$ of all paths on M of class C^2 . In fact, the above two descriptions of $G(T^*M)$ coincide since they use the same Lie algebra action and since $\tilde{P}(T^*M)$ is identified with $T^*P(M)$. For details, we can consult [11] and [7].

4.2 Effects of an 1-cocycle of A on $G(A)$

Let R be an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients of $(A, [\cdot, \cdot], \rho)$, i.e., R is a section of the dual vector bundle $A^* \rightarrow M$ of $A \rightarrow M$ such that, for any $\zeta, \eta \in \Gamma(A)$,

$$\langle [\zeta, \eta], R \rangle = \mathcal{L}_{\rho(\zeta)} \langle \eta, R \rangle - \mathcal{L}_{\rho(\eta)} \langle \zeta, R \rangle,$$

and $\mathbf{ac}^R : \Gamma(A) \rightarrow \Gamma(T(M \times \mathbb{R}))$ the map given, for any $\zeta \in \Gamma(A)$, by

$$\mathbf{ac}^R(\zeta) = \rho(\zeta) + \langle \zeta, R \rangle \frac{\partial}{\partial s},$$

s being the canonical coordinate on the factor \mathbb{R} of $M \times \mathbb{R}$. It is easy to show that \mathbf{ac}^R is an action of A on the fibered manifold $\varpi : \tilde{M} = M \times \mathbb{R} \rightarrow M$ in the sense of [20]. Thus, the vector bundle $\tilde{A} = A \times \mathbb{R} \rightarrow M \times \mathbb{R}$, which is isomorphic [22] to the pull-back bundle $\varpi^*A \rightarrow M \times \mathbb{R}$ of A over ϖ , admits [20] a Lie algebroid structure $([\cdot, \cdot]^R, \rho^R)$. Having identified $\Gamma(\tilde{A})$ with the set of the time-dependent sections of A , the bracket $[\cdot, \cdot]^R$ and the anchor map ρ^R are defined, for any $\tilde{\zeta}, \tilde{\eta} \in \Gamma(\tilde{A})$, respectively, by

$$[\tilde{\zeta}, \tilde{\eta}]^R = [\tilde{\zeta}, \tilde{\eta}] + \langle \tilde{\zeta}, R \rangle \frac{\partial \tilde{\eta}}{\partial s} - \langle \tilde{\eta}, R \rangle \frac{\partial \tilde{\zeta}}{\partial s} \quad \text{and} \quad \rho^R(\tilde{\zeta}) = \rho(\tilde{\zeta}) + \langle \tilde{\zeta}, R \rangle \frac{\partial}{\partial s}.$$

⁷The path $c_1 \odot c_0$ is only piecewise smooth. But, by choosing an appropriate cutoff function $\tau \in C^\infty(\mathbb{R})$, we reparameterize c_i , $i = 0, 1$, by setting $c_i^\tau(t) := \tau'(t)c_i(\tau(t))$. Then, $c_1^\tau \odot c_0^\tau$ is a smooth A -path, homotopic to $c_1 \odot c_0$. For details, see [10].

We denote by $G(\tilde{A})$ the Weinstein groupoid of \tilde{A} . We will examine the relationship which link $G(\tilde{A})$ with $G(A)$.

Any section R of A^* can be integrating over an A -path $c : I \rightarrow A$ by setting

$$\int_c R := \int_0^1 \langle c(t), R(\pi(c(t))) \rangle dt. \quad (35)$$

A basic property of the integral (35) is its invariance under an A -homotopy, if R is an 1-cocycle of A . It is proved in [11] for the 1-cocycles of Lie algebroids coming from Poisson manifolds, i.e., for the Poisson vectors fields, but the same method of proof may be used for the 1-cocycles of any Lie algebroid. Therefore, if R is an 1-cocycle of A and c_0, c_1 are two homotopic A -paths, we have

$$\int_{c_0} R = \int_{c_1} R.$$

From the above property we deduce that the map $r : P(A) \rightarrow \mathbb{R}$, $c \mapsto \int_c R$, descends to a well defined map, also denoted by r , on the quotient space $G(A)$ by setting

$$r([c]) := \int_c R.$$

The additivity of integration with respect the concatenation of paths shows that r is a multiplicative function on $G(A)$, i.e., for any $([c_1], [c_0]) \in G_2(A)$,

$$r([c_1] \cdot [c_0]) = r([c_1]) + r([c_0]).$$

Hence, we can construct the action groupoid $\widetilde{G(A)} \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} \tilde{M}$ associated to the left action $\mathbf{a}c^r$ of $G(A)$ on $\varpi : \tilde{M} = M \times \mathbb{R} \rightarrow M$ given by (19). We have

Proposition 4.5 ([13]) (i) *The topological groupoids $\widetilde{G(A)}$ and $G(\tilde{A})$ are isomorphic.*
(ii) *A is integrable if and only if \tilde{A} is. In this case, the previous isomorphism is a Lie groupoid isomorphism.*

4.3 Integration of twisted Jacobi structures

Let (M, Λ, E, ω) be a twisted Jacobi manifold, $(T^*M \times \mathbb{R}, \{\cdot, \cdot\}^\omega, \pi \circ (\Lambda, E)^\#, (-E, 0))$ its associated Lie algebroid with 1-cocycle and $G(T^*M \times \mathbb{R})$ the Weinstein groupoid of $T^*M \times \mathbb{R}$. Let, also, $(\tilde{M}, \tilde{\Lambda}, \tilde{\omega})$ be the twisted Poissonization of (M, Λ, E, ω) , i.e., $\tilde{M} = M \times \mathbb{R}$, $\tilde{\Lambda} = e^{-s}(\Lambda + \frac{\partial}{\partial s} \wedge E)$ and $\tilde{\omega} = e^s \omega$, $(\langle \cdot, \cdot \rangle^{d\tilde{\omega}}, \tilde{\Lambda}^\#)$ the corresponding Lie algebroid structure on $T^*\tilde{M}$, and $G(T^*\tilde{M})$ the Weinstein groupoid of $T^*\tilde{M}$. By applying the results of the previous subsection to the case where (A, R) is the Lie algebroid with 1-cocycle $(T^*M \times \mathbb{R}, (-E, 0))$ and taking into account that $(T^*M \times \mathbb{R}) \times \mathbb{R} \cong T^*(M \times \mathbb{R})$, we get, as direct consequence of Proposition 4.5, the following

Proposition 4.6 (i) *There is an isomorphism of topological groupoids*

$$G(T^*M \times \mathbb{R}) \cong G(T^*\tilde{M}). \quad (36)$$

(ii) *The Lie algebroid $T^*M \times \mathbb{R}$ is integrable if and only if $T^*\tilde{M}$ is.*

We now establish, via Proposition 3.5, the inverse of Theorem 3.3.

Theorem 4.7 *An integrable twisted Jacobi manifold (M, Λ, E, ω) is integrated by a twisted contact groupoid.*

Proof. The assumption (M, Λ, E, ω) is integrable means that the corresponding Lie algebroid $(T^*M \times \mathbb{R}, \{\cdot, \cdot\}^\omega, \pi \circ (\Lambda, E)^\#)$ is integrable. Consequently, according to Proposition 4.6, $(T^*\tilde{M}, [\cdot, \cdot]^{d\tilde{\omega}}, \tilde{\Lambda}^\#)$ is also integrable; fact which implies that $(\tilde{M}, \tilde{\Lambda}, \tilde{\omega}, \frac{\partial}{\partial s})$ is integrable as homogeneous twisted Poisson manifold. Crainic's and Fernandes's Theorem show that $T^*M \times \mathbb{R}$ (resp. $T^*\tilde{M}$) is integrated by $G(T^*M \times \mathbb{R}) \xrightarrow[\beta]{\alpha} M$ (resp.

$G(T^*\tilde{M})$) which is the unique target-simply connected Lie groupoid integrating it. Furthermore, Cattaneo's and Xu's Theorem in [6] assures us that $G(T^*\tilde{M})$ is endowed with a non-degenerate, multiplicative, twisted symplectic form $\tilde{\Omega}$. Because of (36), it induces on $\widetilde{G(T^*M \times \mathbb{R})} \xrightarrow[\tilde{\beta}]{\tilde{\alpha}} \tilde{M}$ a non-degenerate, multiplicative, $(\tilde{\alpha}^*d\tilde{\omega} - \tilde{\beta}^*d\tilde{\omega})$ -twisted symplectic form denoted, also, by $\tilde{\Omega}$. From construction, $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$ is a multiplicative vector field on $\widetilde{G(T^*M \times \mathbb{R})} = G(T^*M \times \mathbb{R}) \times \mathbb{R}$ whose the corresponding vector field $\frac{\partial}{\partial s}$ on the base manifold $\tilde{M} = M \times \mathbb{R}$ is the homothety vector field of the $d\tilde{\omega}$ -twisted Poisson structure $\tilde{\Lambda}$. We need to show that $\tilde{\Omega}$ is homogeneous with respect to $\frac{\partial}{\partial s}$.

The multiplicativity of $\frac{\partial}{\partial s}$ on $\widetilde{G(T^*M \times \mathbb{R})} = G(T^*M \times \mathbb{R}) \times \mathbb{R}$ implies that its flows

$$\begin{aligned} \Psi_u : G(T^*M \times \mathbb{R}) \times \mathbb{R} &\longrightarrow G(T^*M \times \mathbb{R}) \times \mathbb{R} \\ (g, s) &\longrightarrow (g, s + u) \end{aligned}$$

are (local) Lie groupoid automorphisms over the transformations $\psi_u : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ defined by $\psi_u(x, s) = (x, s + u)$. Thus, $\tilde{\alpha} \circ \Psi_u^{-1} = \psi_u^{-1} \circ \tilde{\alpha}$. Let $\tilde{\Lambda}_G$ be the $(\tilde{\alpha}^*d\tilde{\omega} - \tilde{\beta}^*d\tilde{\omega})$ -twisted Poisson structure on $\widetilde{G(T^*M \times \mathbb{R})}$ defined by the inversion of $\tilde{\Omega}$. From Theorem 2.6 of [6], we have that $\tilde{\Lambda}_G$ is projectable on \tilde{M} via $\tilde{\alpha}_*$ and its projection is the $d\tilde{\omega}$ -twisted Poisson structure $\tilde{\Lambda}$ which is homogeneous with respect to $\frac{\partial}{\partial s}$. On the other hand, we have $\psi_u^*\tilde{\Lambda} = (\psi_u^{-1})_*\tilde{\Lambda} = e^{-u}\tilde{\Lambda}$. Consequently, $(\tilde{\alpha}_* \circ (\Psi_u^{-1})_*)\tilde{\Lambda}_G = ((\psi_u^{-1})_* \circ \tilde{\alpha}_*)\tilde{\Lambda}_G \Leftrightarrow \tilde{\alpha}_*((\Psi_u^{-1})_*\tilde{\Lambda}_G) = e^{-u}\tilde{\Lambda}$, whence, taking into account the expression of Ψ_u , we obtain $(\Psi_u^{-1})_*\tilde{\Lambda}_G = e^{-u}\tilde{\Lambda}_G$. Hence, $\mathcal{L}_{\frac{\partial}{\partial s}}\tilde{\Lambda}_G = \frac{d}{du}(\Psi_u^*\tilde{\Lambda}_G)|_{u=0} = \frac{d}{du}(e^{-u}\tilde{\Lambda}_G)|_{u=0} = -\tilde{\Lambda}_G$, which means that $\tilde{\Lambda}_G$ is homogeneous with respect to $\frac{\partial}{\partial s}$. For this, its inverse $\tilde{\Omega}$ is also homogeneous with respect to $\frac{\partial}{\partial s}$, i.e., $\mathcal{L}_{\frac{\partial}{\partial s}}\tilde{\Omega} = \tilde{\Omega}$.

We remark that $\tilde{\Omega}$, as an exact twisted symplectic form, can be written as $\tilde{\Omega} = \tilde{\alpha}^*\tilde{\omega} - \tilde{\beta}^*\tilde{\omega} + d\tilde{\eta}$, where $\tilde{\eta}$ is an 1-form on $\widetilde{G(T^*M \times \mathbb{R})}$ such that $d\tilde{\eta}$ is multiplicative. Since the part $\tilde{\alpha}^*\tilde{\omega} - \tilde{\beta}^*\tilde{\omega}$ of $\tilde{\Omega}$ is homogeneous with respect to $\frac{\partial}{\partial s}$, it is clear that the homogeneity of $\tilde{\Omega}$ implies that of $d\tilde{\eta}$, i.e., $\mathcal{L}_{\frac{\partial}{\partial s}}d\tilde{\eta} = d\tilde{\eta} \Leftrightarrow d(i(\frac{\partial}{\partial s})d\tilde{\eta}) = d\tilde{\eta}$. We set $\tilde{\vartheta} = i(\frac{\partial}{\partial s})d\tilde{\eta}$ and we check that $\mathcal{L}_{\frac{\partial}{\partial s}}\tilde{\vartheta} = \tilde{\vartheta}$. So, we have $\tilde{\vartheta} = e^s\vartheta$, where ϑ is a 1-form on $G(T^*M \times \mathbb{R})$, and $d\tilde{\vartheta} = d\tilde{\eta}$. Therefore, $\tilde{\Omega} = \tilde{\alpha}^*\tilde{\omega} - \tilde{\beta}^*\tilde{\omega} + d\tilde{\vartheta} = e^s(\alpha^*\omega - e^{-r}\beta^*\omega) + d(e^s\vartheta)$. According to Proposition 3.5, we conclude that $(\vartheta, \alpha^*\omega - e^{-r}\beta^*\omega)$ defines a twisted contact structure on $G(T^*M \times \mathbb{R})$ whose induced twisted Jacobi structure on the manifolds of units is the initial given one (see, Proposition 3.7) \blacklozenge

References

[1] Albert, C., Un theoreme de realisation des varietes de Jacobi, C. R. Acad. Sci. Paris Ser. I Math. 317 (1993), 77–80.

- [2] Almeida, R., Molino, P., Suites d'Atiyah et feuilletages transversalement complets, *C. R. Acad. Sci. Paris Ser. I Math.* 300 (1985), 13–15.
- [3] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, Ch., Integration of twisted Dirac brackets, *Duke Math. J.* 123 (2004), 549–607.
- [4] Cannas da Silva, A., Weinstein, A., Geometric Models for Noncommutative Algebras, A.M.S., Berkeley Mathematics Lecture Notes Series, Providence, 1999.
- [5] Cattaneo, A.S., Felder, G., Poisson sigma models and symplectic groupoids, in: Landsman, N.P., Pflaum, M., Schlichenmeier, M. (Eds.), *Quantization of Singular Symplectic Quotients*, Progr. Math. 198, Birkhauser, Basel, 2001, 61–93.
- [6] Cattaneo, A.S., Xu, P., Integration of twisted Poisson structures, *J. Geom. Phys.* 49 (2004), 187–196.
- [7] Cattaneo, A.S., On the integration of Poisson manifolds, Lie algebroids, and coisotropic submanifolds, *Lett. Math. Phys.* 67 (2004), 33–48.
- [8] Chinea, D., Marrero, J.C., de Leon, M., Prequantizable Poisson manifolds and Jacobi structures, *J. Phys. A* 29 (1996), 6313–6324.
- [9] Coste, A., Dazord, P., Weinstein, A., Groupoïdes symplectiques, *Publications du Département de Mathématiques. Nouvelle Série. A*, Vol. 2 (1987), Publ. Dep. Math. Nouvelle Ser. A, Univ. Claude-Bernard, Lyon, 1–62.
- [10] Crainic, M., Fernandes, R.L., Integrability of Lie brackets, *Ann. of Math.* 157 (2003), 575–620.
- [11] Crainic, M., Fernandes, R.L., Integrability of Poisson brackets, *J. Differential Geom.* 66 (2004), 71–137.
- [12] Crainic, M., Fernandes, R.L., Rigidity and flexibility in Poisson geometry, *Trav. Math.*, XVI (2005), Univ. Luxemb., 53–68.
- [13] Crainic, M., Zhu, Ch., Integrability of Jacobi and Poisson structures, *Ann. Inst. Fourier (Grenoble)* 57 (2007), 1181–1216.
- [14] Dazord, P., Groupoïdes symplectiques et troisième théorème de Lie "non linéaire", in: Albert, C. (Ed), *Géométrie Symplectique et Mécanique* (La Grande Motte, France 1988), Lecture Notes in Math., 1416, Springer, Berlin, 1990, 39–74.
- [15] Dazord, P., Hector, G., Intégration symplectique des variétés de Poisson totalement asphériques, in: Dazord, P., Weinstein, A. (Eds), *Symplectic geometry, groupoids, and integrable systems* (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., 20, Springer, New York, 1991, 37–72.
- [16] Dazord, P., Lichnerowicz, A., Marle, Ch.-M., Structure locale des variétés de Jacobi, *J. Math. Pures Appl.* 70 (1991), 101–152.
- [17] Dazord, P., Sur l'intégration des algèbres de Lie locales et la préquantification, *Bull. Sci. Math.* 121 (1997), 423–462.
- [18] Dufour, J.-P., Zung, N. T., Poisson structures and their normal forms, *Progress in Mathematics* 242, Birkhauser Verlag, Basel, 2005.

- [19] Guedira, F., Lichnerowicz, A., Géométrie des algèbres de Lie locales de Kirillov, *J. Math. Pures Appl.* 63 (1984), 407–484.
- [20] Higgins, Ph. J., Mackenzie, K., Algebraic constructions in the category of Lie algebroids, *J. Algebra* 129 (1990), 194–230.
- [21] Hochgerner, S., Garcia-Naranjo, L., *G*-Chaplygin systems with internal symmetries, truncation, and an (almost) symplectic view of Chaplygin’s ball, *J. Geom. Mech.* 1 (2009), 35–53.
- [22] Iglesias, D., Marrero, J.C., Generalized Lie bialgebroids and Jacobi structures, *J. Geom. Phys.* 40 (2001), 176–200.
- [23] Kerbrat, Y., Souici-Benhammadi, Z., Variétés de Jacobi et groupoïdes de contact, *C. R. Acad. Sci. Paris, Série I*, 317 (1993), 81–86.
- [24] Klimčík, C., Ströbl, T., WZW-Poisson manifolds, *J. Geom. Phys.* 43 (2002), 341–344.
- [25] Libermann, P., Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact, *Colloque Géom. Diff. Globale (Bruxelles, 1958)*, Centre Belge Rech. Math. Louvain 1959, 37–59.
- [26] Libermann, P., Sur quelques exemples de structures pfaffiennes et presque cosymplectiques, *Ann. Mat. Pura Appl.* 60 (1962), 153–172.
- [27] Libermann, P., On symplectic and contact groupoids, *Differential geometry and its applications, (Opava 1992)*, Math. Publ., 1, Silesian Univ., Opava, 1993, 29–45.
- [28] Libermann, P., Marle, Ch.-M., *Symplectic geometry and analytical mechanics, Mathematics and its Applications*, 35. D. Reidel Publishing Co., Dordrecht, 1987.
- [29] Lichnerowicz, A., Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. pures et appl.* 57 (1978), 453–488.
- [30] Mackenzie, K., *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, 124, Cambridge University Press, Cambridge, 1987.
- [31] Mackenzie, K., *General theory of Lie groupoids and Lie algebroids*, London Mathematical Society Lecture Note Series, 213, Cambridge University Press, Cambridge, 2005.
- [32] Marle, Ch.-M., Calculus on Lie algebroids, Lie groupoids and Poisson manifolds, *Dissertationes Math. (Rozprawy Mat.)* 457 (2008), 57 pp.
- [33] Nunes da Costa, J.M., Petalidou, F., Twisted Jacobi manifolds, twisted Dirac-Jacobi structures and quasi-Jacobi bialgebroids, *J. Phys. A: Math. Gen.* 39 (2006), 10449–10475.
- [34] Nunes da Costa, J.M., Petalidou, F., Characteristic foliation of twisted Jacobi manifolds, in: Iglesias Ponte, D. *et al.*, *Proceedings of XV Fall Workshop on Geometry and Physics, Tenerife, 2006*, Publ. de la RSME 11 (2007), 322–327.
- [35] Park, J.S., Topological open p -branes, in: Fukaya, K., Oh, Y.G., Ono, K., Tian, G. (Eds), *Symplectic geometry and Mirror Symmetry (Seoul, 2000)*, World Sci. Publishing, River Edge, NJ, 2001, 311–384.

- [36] Pradines, J., Troisième théorème de Lie des groupoïdes différentiables, *C. R. Acad. Sci. Paris Ser. A-B* 267 (1968), A21–A23.
- [37] Ratiu, T., Schmid, R., The differentiable structure of three remarkable diffeomorphism groups, *Math. Z.* 177 (1981) 81–100.
- [38] Roytenberg, D., Quasi-Lie bialgebroids and twisted Poisson manifolds, *Lett. Math. Phys.* 61 (2002), 123–137.
- [39] Ševera, P., Weinstein, A., Poisson geometry with a 3-form background, in: Maeda, Y., Watamura, S. (Eds), *Noncommutative Geometry and String Theory*, *Prog. Theor. Phys. Suppl.* 144 (2001), 145–154.
- [40] Weinstein, A., Symplectic groupoids and Poisson manifolds, *Bull. Amer. Math. Soc. (N.S.)* 16 (1987), 101–104.
- [41] Weinstein, A., Blowing up realizations of Heisenberg-Poisson manifolds, *Bull. Sci. Math.* 113 (1989), 381–406.
- [42] Zambon, M., Zhu, Ch., On the geometry of prequantization spaces, *J. Geom. Phys.* 57 (2007), 2372–2397.